Odine degli Ingegneri della Provincia di Pistoia
Corso sulla Vulnerabilità Sismica

Modelli evolutivi per la verifica del rischio di edifici esistenti

Quaderno 4
Primi concetti di analisi nonlineare

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DIPARTIMENTO DI INGEGNERIA E GEOLOGIA

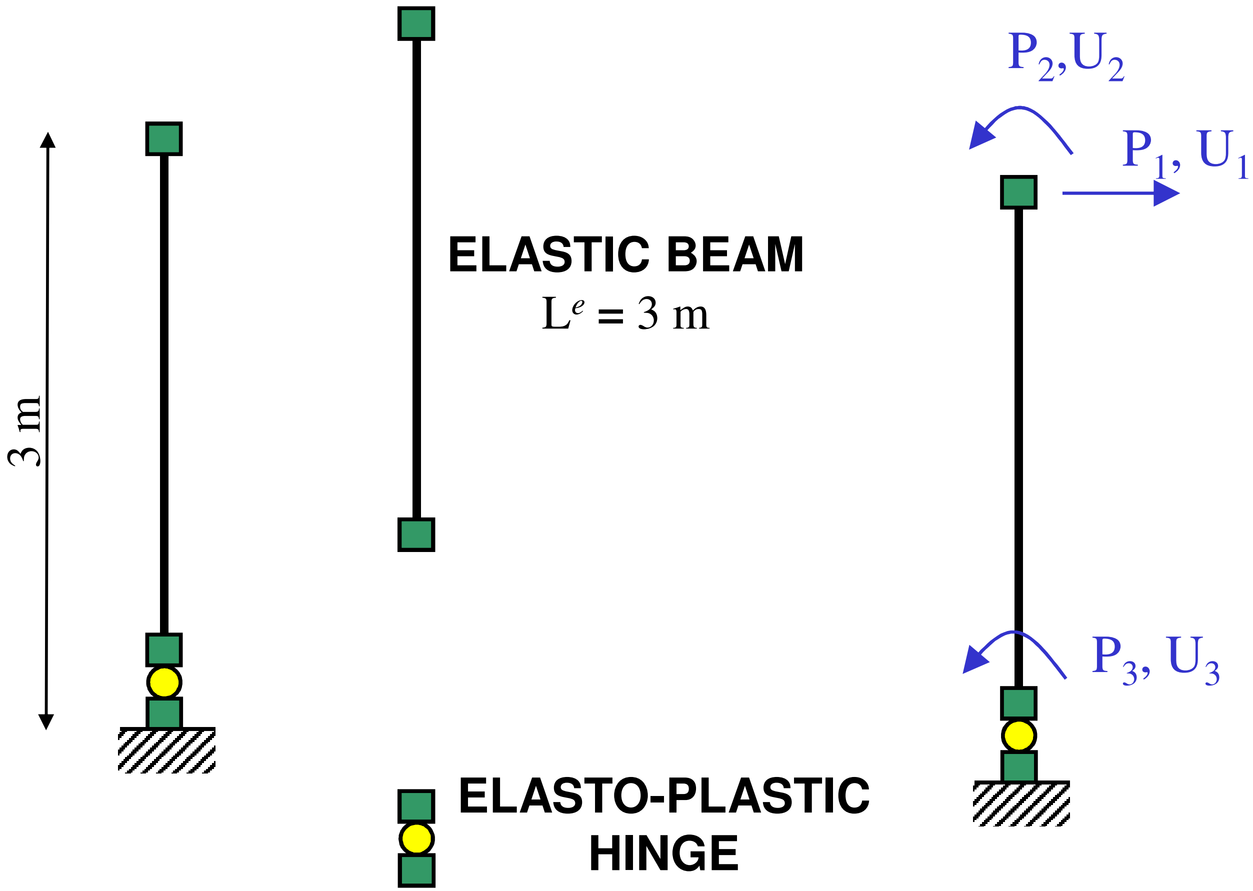
31 Maggio 2013

Ud'A

INTRODUCTION

- Three examples are presented hereafter to introduce nonlinear problems and nonlinear solution schemes

Example 1: Material Nonlinearity

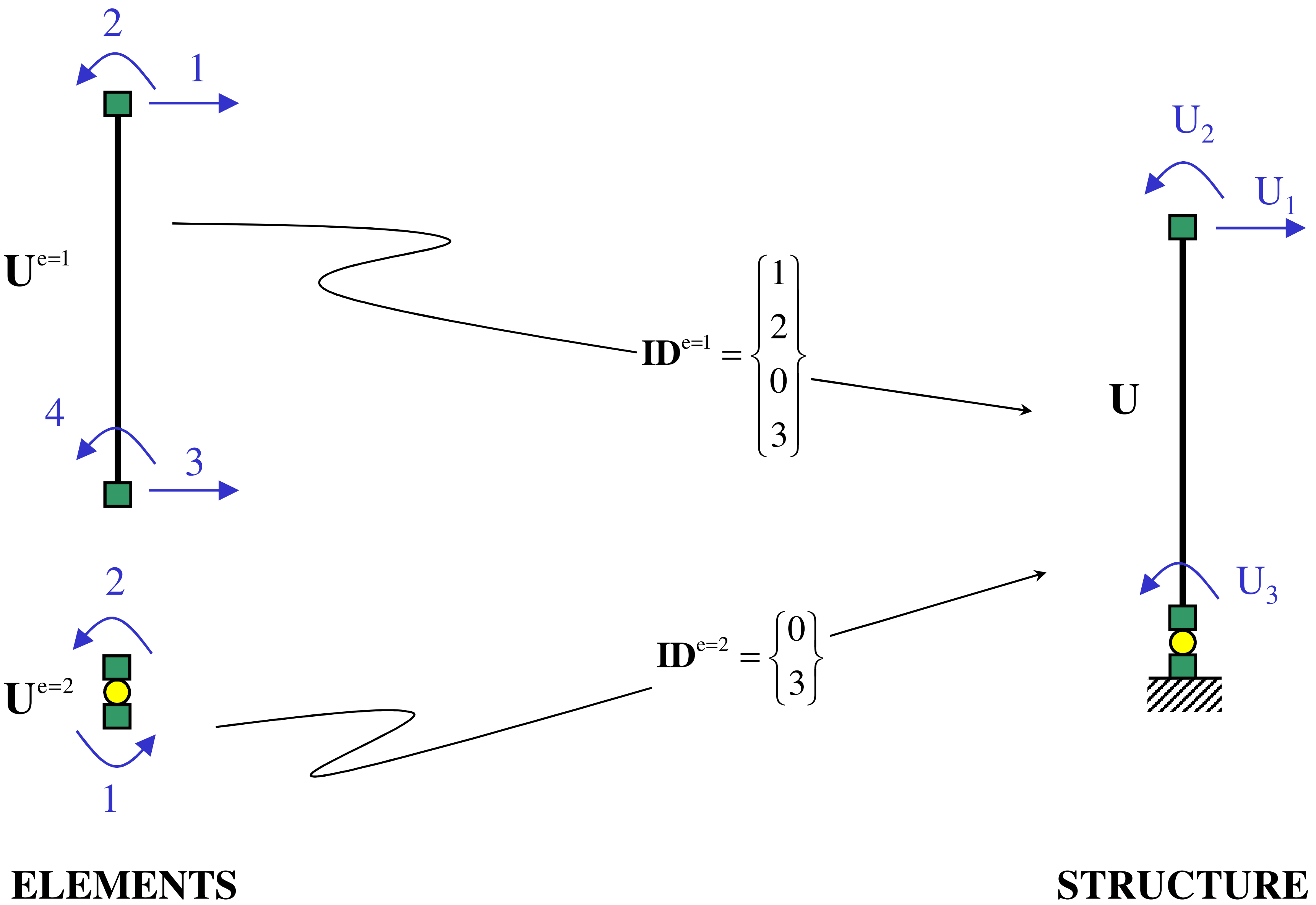


$$\mathbf{P} = \mathbf{KU}$$

$$\mathbf{P} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

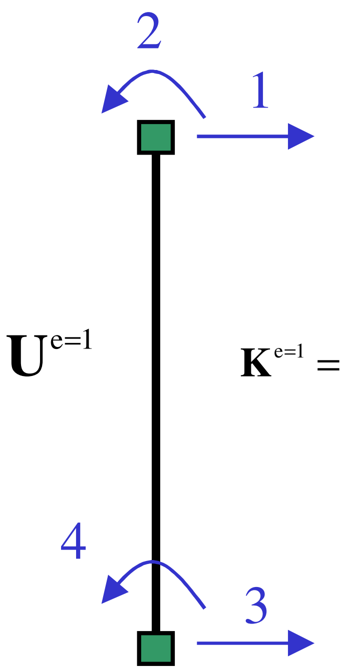
$$\mathbf{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

Example 1



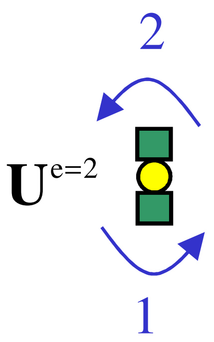
2
1
U^{e=1}
4
3
2
U^{e=2}
1
ID^{e=1} = {1 2 0 3}
ID^{e=2} = {0 3}
U_2
U_1
U
U_3
ELEMENTS
STRUCTURE

Example 1



Columns and rows: U_1, U_2, U_3

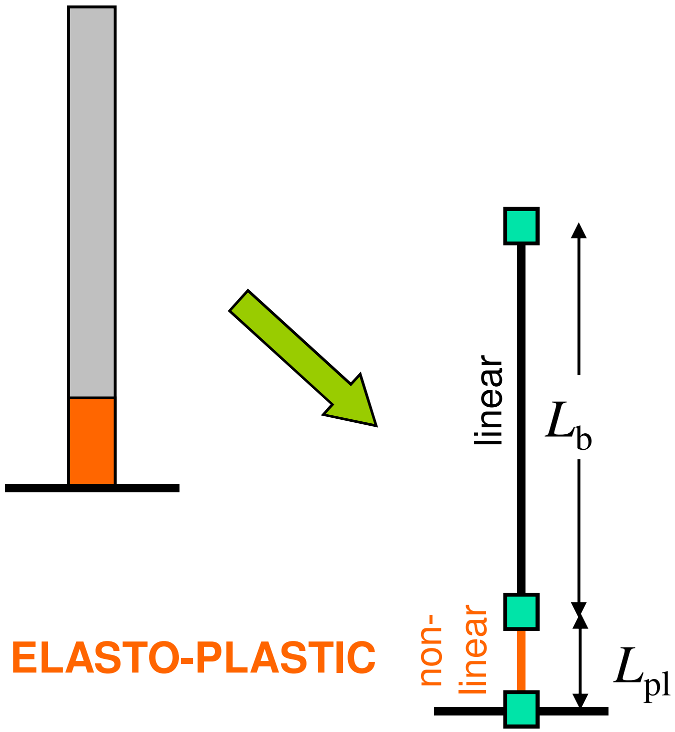
$$K^{e=1} = EI_b \begin{bmatrix} \frac{12}{L_b^3} & \frac{6}{L_b^2} & -\frac{12}{L_b^3} & \frac{6}{L_b^2} \\ \frac{6}{L_b^2} & \frac{4}{L_b} & -\frac{6}{L_b^2} & \frac{2}{L_b} \\ -\frac{12}{L_b^3} & -\frac{6}{L_b^2} & \frac{12}{L_b^3} & -\frac{6}{L_b^2} \\ \frac{6}{L_b^2} & \frac{2}{L_b} & -\frac{6}{L_b^2} & \frac{4}{L_b} \end{bmatrix}$$

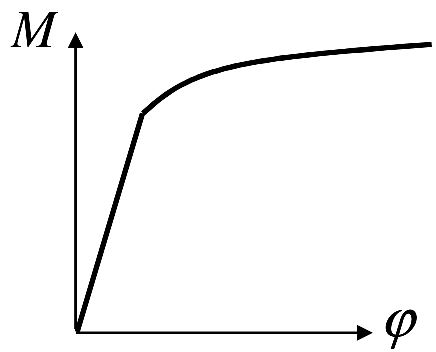


$$K^{e=2} = k_h \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow U_3$$

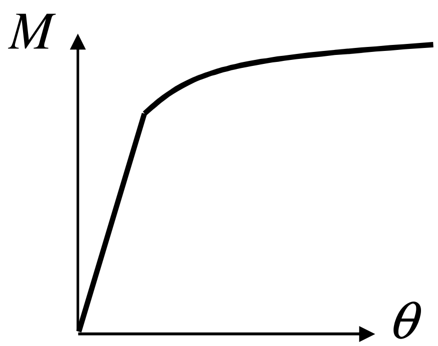
$$K = EI_b \begin{bmatrix} \frac{12}{L_b^3} & \frac{6}{L_b^2} & \frac{6}{L_b^2} \\ \frac{6}{L_b^2} & \frac{4}{L_b} & \frac{2}{L_b} \\ \frac{6}{L_b^2} & \frac{2}{L_b} & \frac{4}{L_b} + k_h \end{bmatrix}$$

Example 1





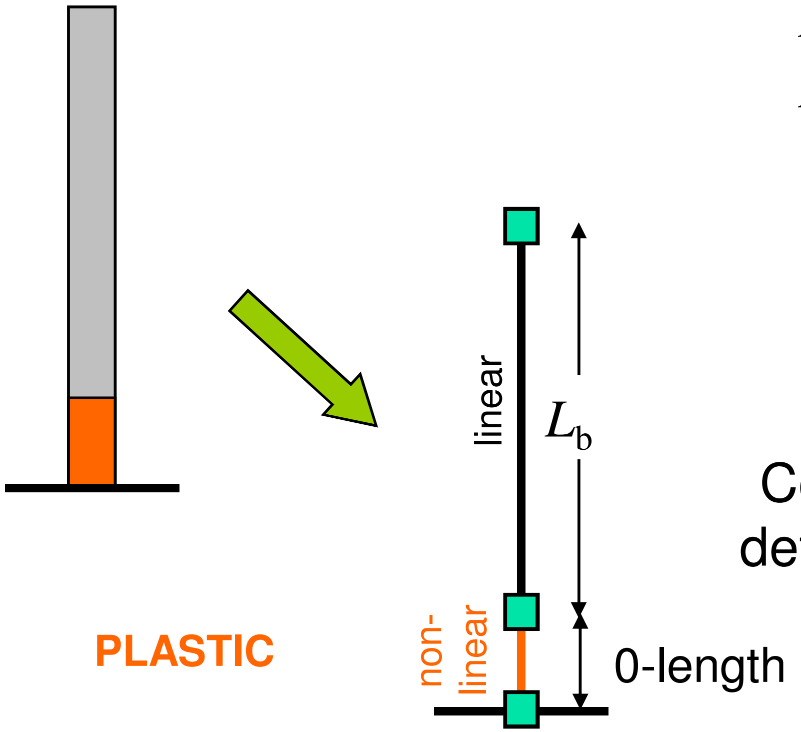
Hp:
curvature φ
constant over L_{pl} → $\theta = \varphi L_{pl}$

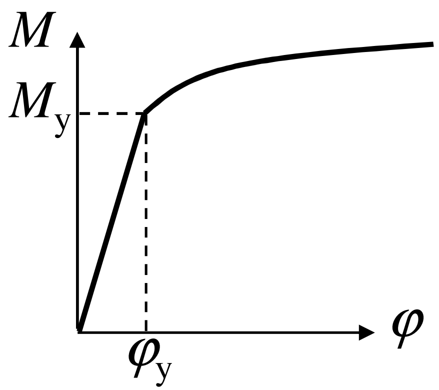


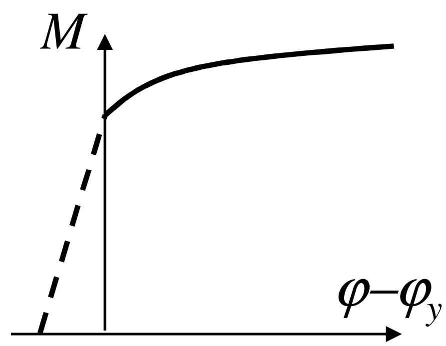
Alternative 1

Order of magnitude of plastic hinge length $\frac{d}{2} \leq L_{pl} \leq d$

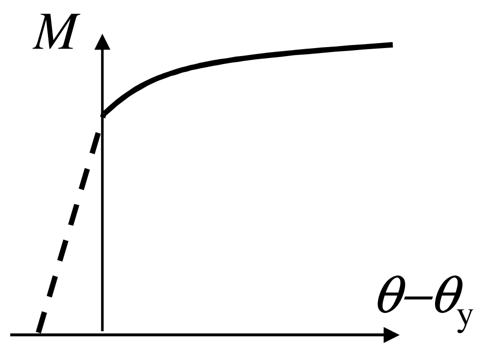
Example 1





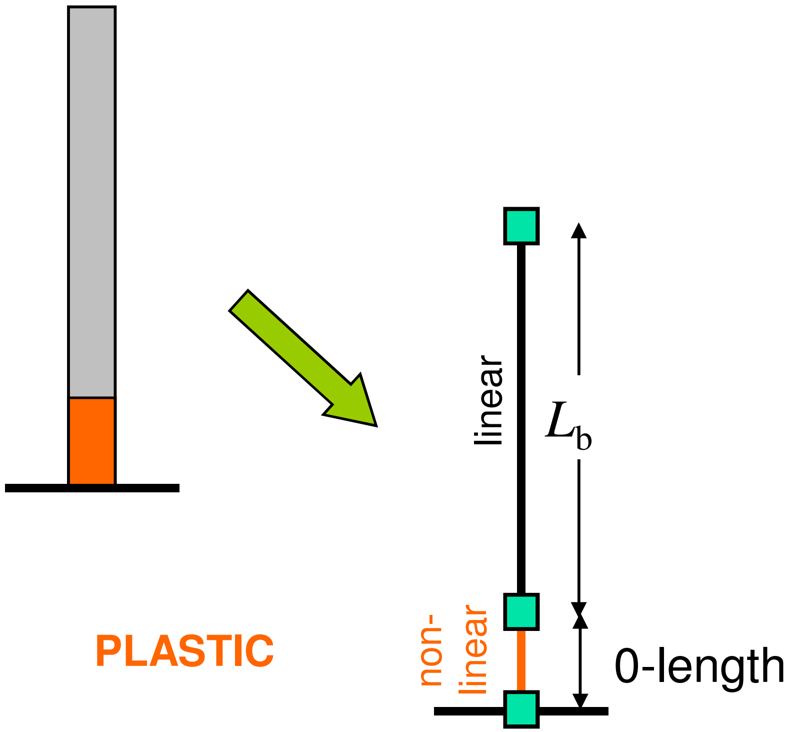


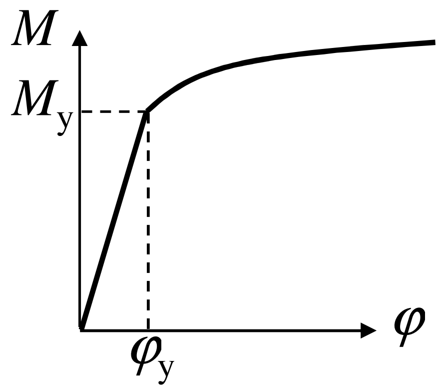
Consider plastic deformations only → $(\theta - \theta_y) = (\varphi - \varphi_y) L_{pl}$



Alternative 2

Example 1



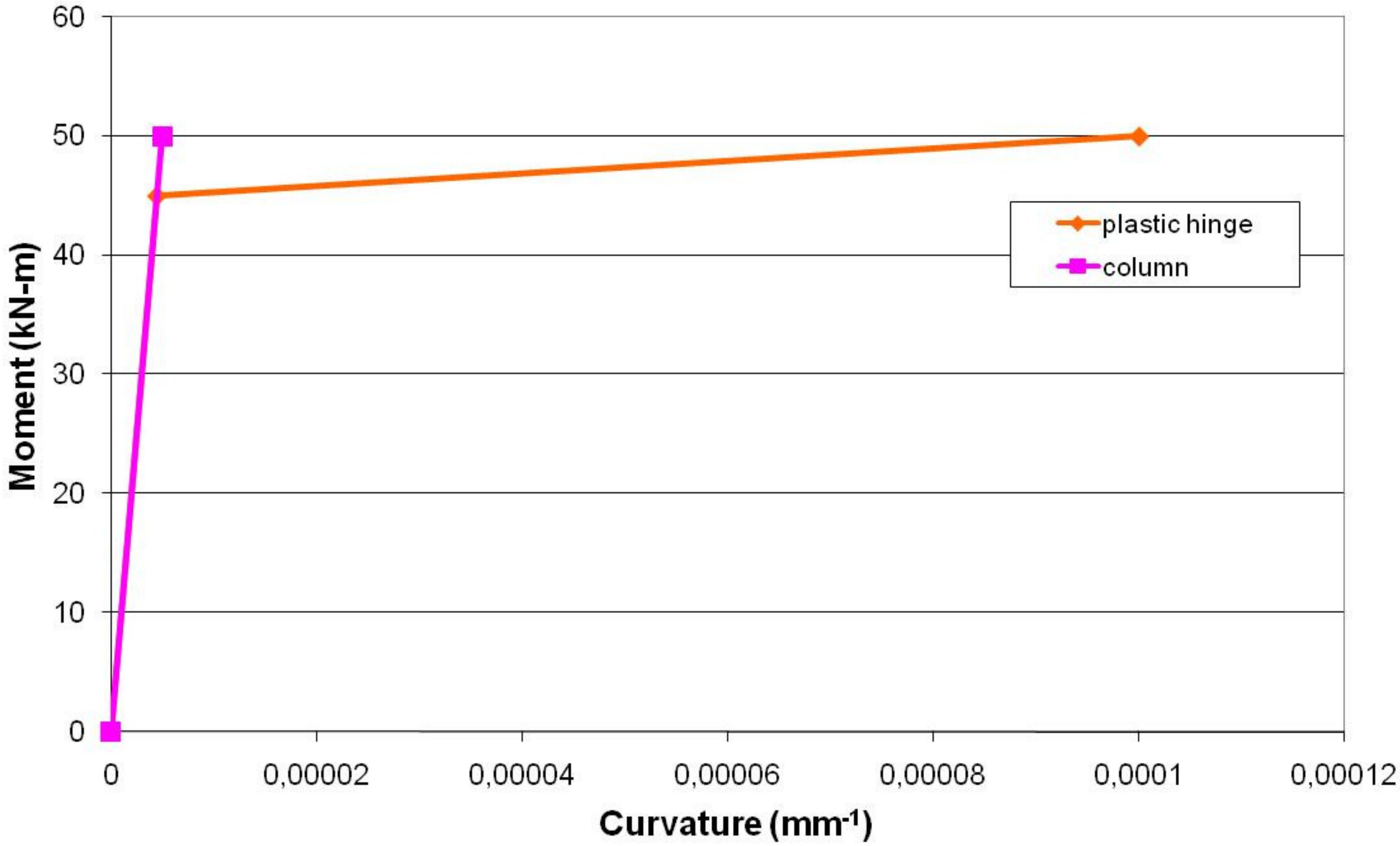


Procedure followed here

Strictly speaking this is not correct
The flexibility of the plastic hinge length is accounted for twice, both in the column element and in the hinge

This approach is followed to illustrate the nonlinear procedure

Example 1



$L_{pl} = 200$ mm

$EI_{el\text{-}b} = 10^{13}$ N-mm^2

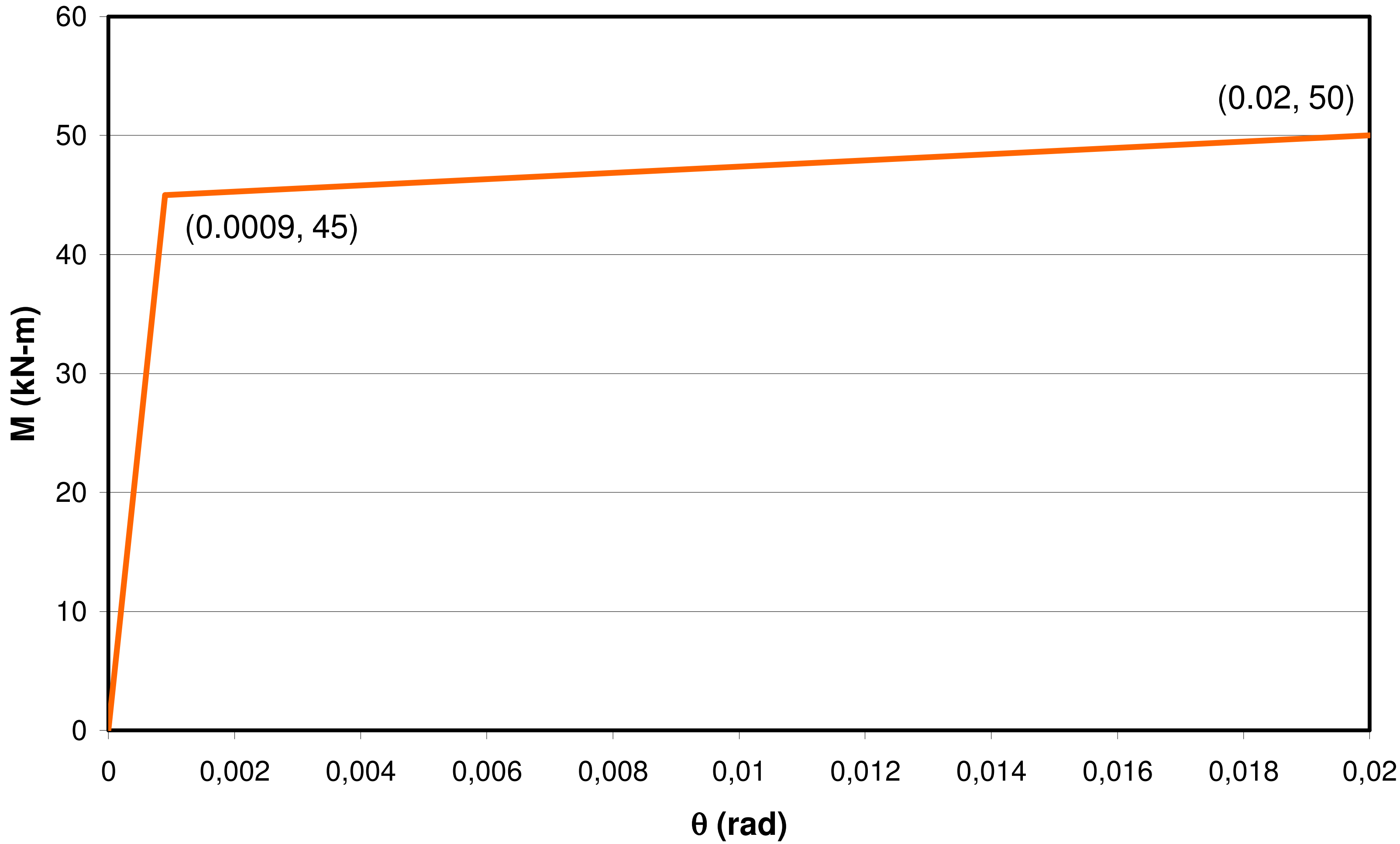
$EI_{el\text{-}h} = 10^{13}$ N-mm^2

$EI_{pl\text{-}h} = 5{,}2\text{x}10^{10}$ N-mm^2

$k_{el\text{-}h} = EI_{el\text{-}cp}/L_{pl} = 5\ \text{x}10^{10}$ N-mm

$k_{pl\text{-}h} = EI_{pl\text{-}cp}/L_{pl} = 2{,}6\text{x}10^{8}$ N-mm

Example 1

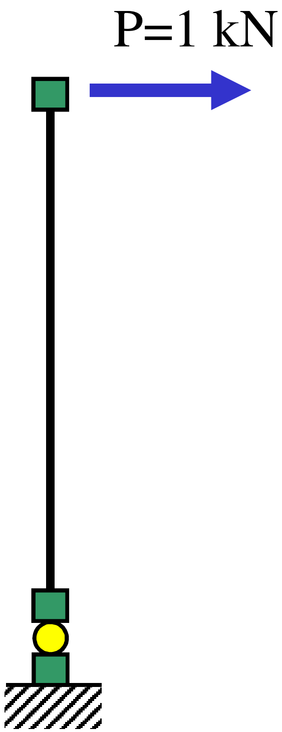


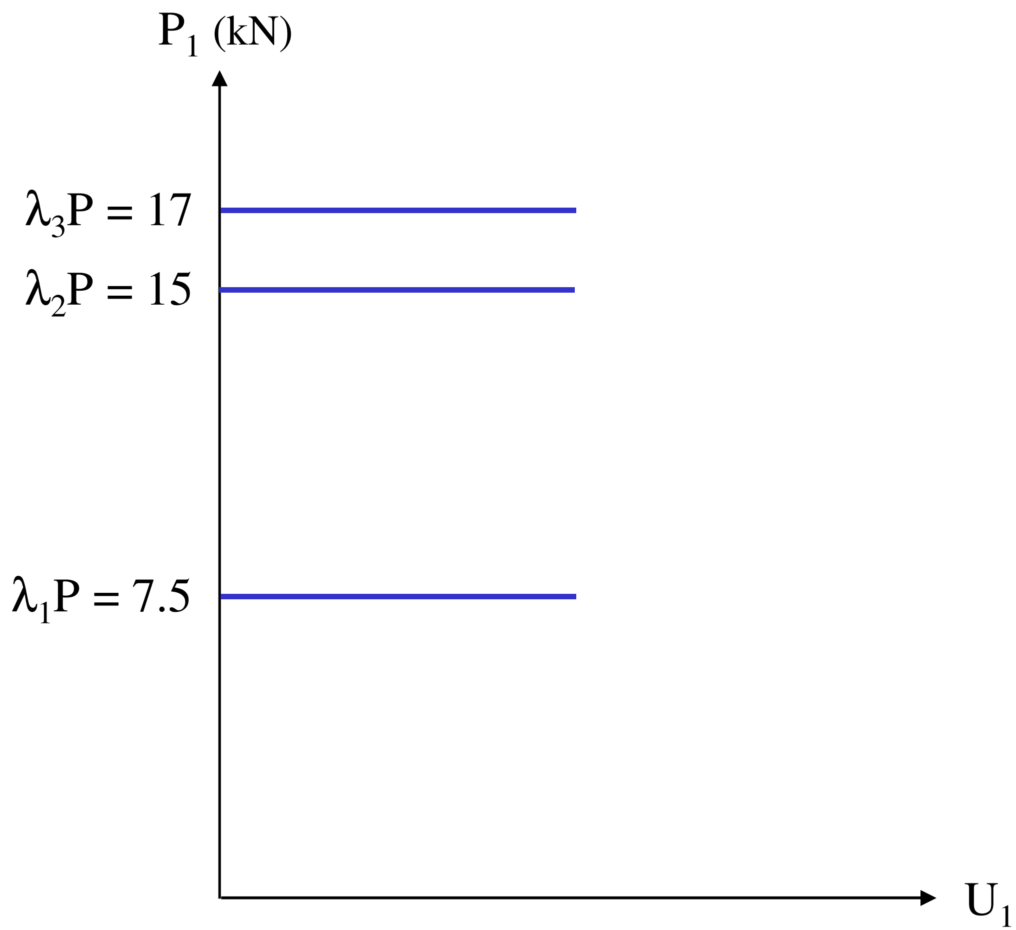
$EI_{el\text{-}b} = 10^4$ kN-m^2

$k_{el\text{-}h} = 5 \times 10^4$ kN-m

$k_{pl\text{-}h} = 2{,}6 \times 10^2$ kN-m

Example 1





Load History

$P_1 = \lambda P$

$\lambda = \{7.5, 15, 17\}$

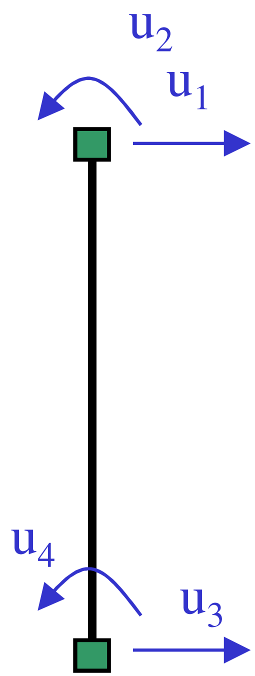
Example 1

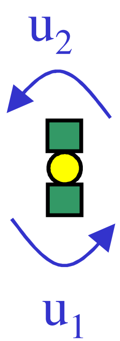
nodal displ.s

$$\mathbf{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

nodal forces

$$\mathbf{P} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$





plastic hinge rotation

$$\theta_h = u_2 - u_1 = u_2$$

Example 1

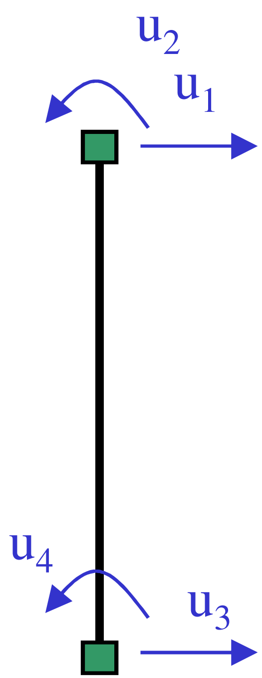
LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

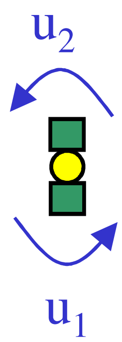
i**=1**

$$\mathbf{U} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{P}_{tr} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{P}_h = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{P}_R = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{P} = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$

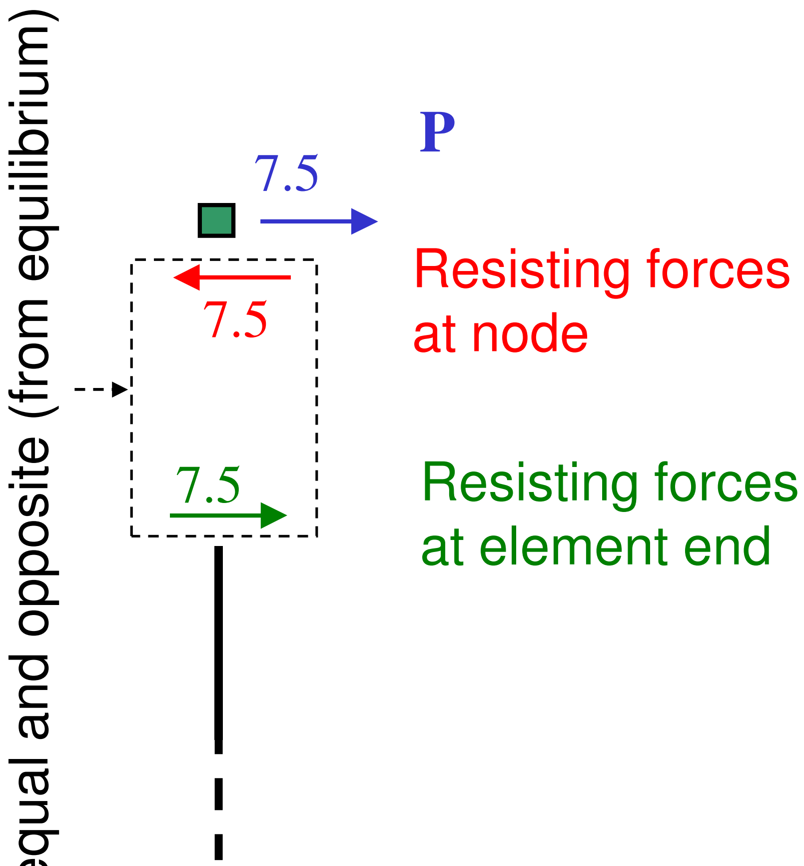
$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$





Example 1

SIGN CONVENTION: When equilibrium is reached:



$$\mathbf{P} + \mathbf{P}_R = \mathbf{0}$$

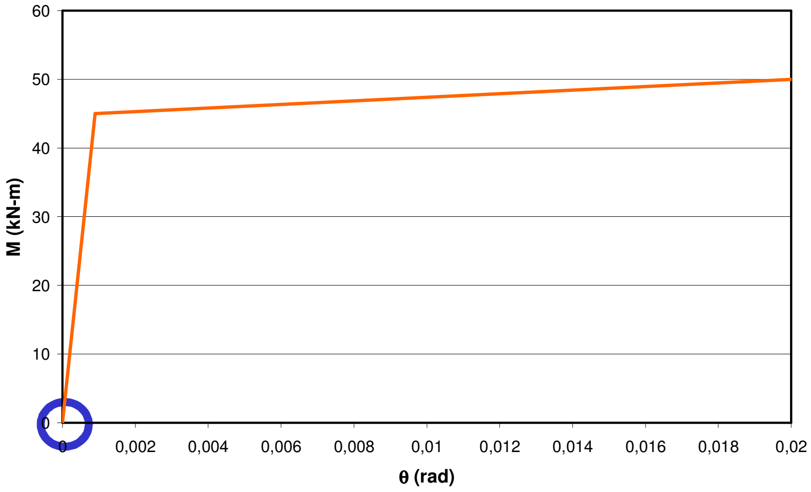
$$\mathbf{P} - \mathbf{P}_R = \mathbf{0}$$

Convention used here:
formally less correct
easier to represent

Example 1

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

***i*=1**



Initial stiffness

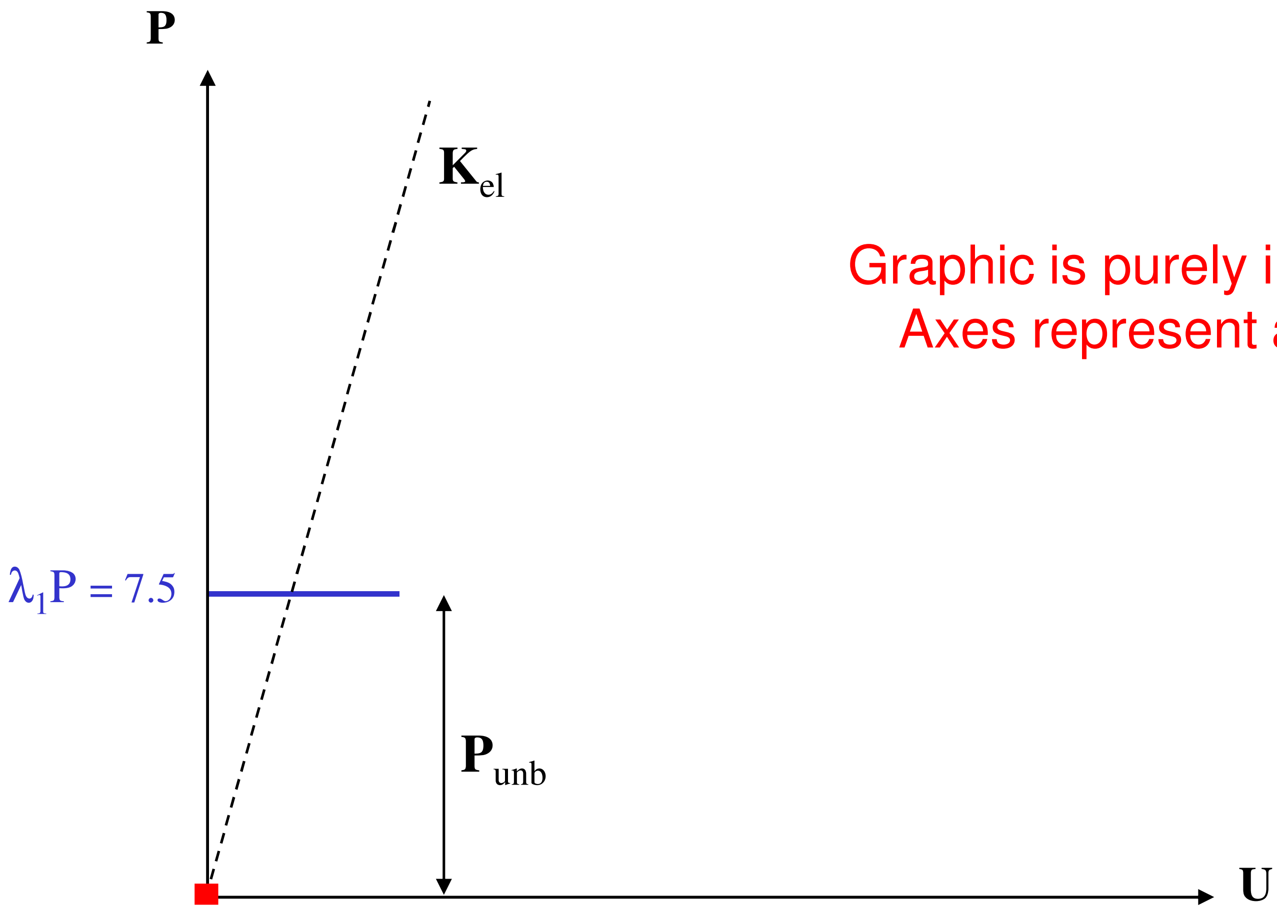
$EI_b = 10^4\ \text{kN-m}^2$

$k_h = k_{el\text{-}h} = EI_{el\text{-}h}/L_{pl} = 5\ \text{x}10^4\ \text{kN-m}$

$L_b = 3\ \text{m}$

$$\mathbf{K} = \mathbf{K}_{el} = \begin{bmatrix} \frac{12EI_b}{L_b^3} & \frac{6EI_b}{L_b^2} & \frac{6EI_b}{L_b^2} \\ \frac{6EI_b}{L_b^2} & \frac{4EI_b}{L_b} & \frac{2EI_b}{L_b} \\ \frac{6EI_b}{L_b^2} & \frac{2EI_b}{L_b} & \frac{4EI_b}{L_b} + k_{el-h} \end{bmatrix}$$

Example 1

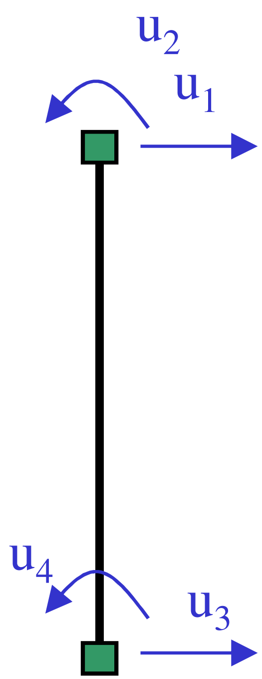


Graphic is purely indicative
Axes represent arrays!

Example 1

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

***i*=1**



$$\Delta\mathbf{U} = \mathbf{K}^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix}$$

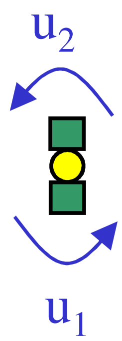
$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix}$$

$$\mathbf{U}_b = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ 0 \\ -0.00045 \end{Bmatrix}$$

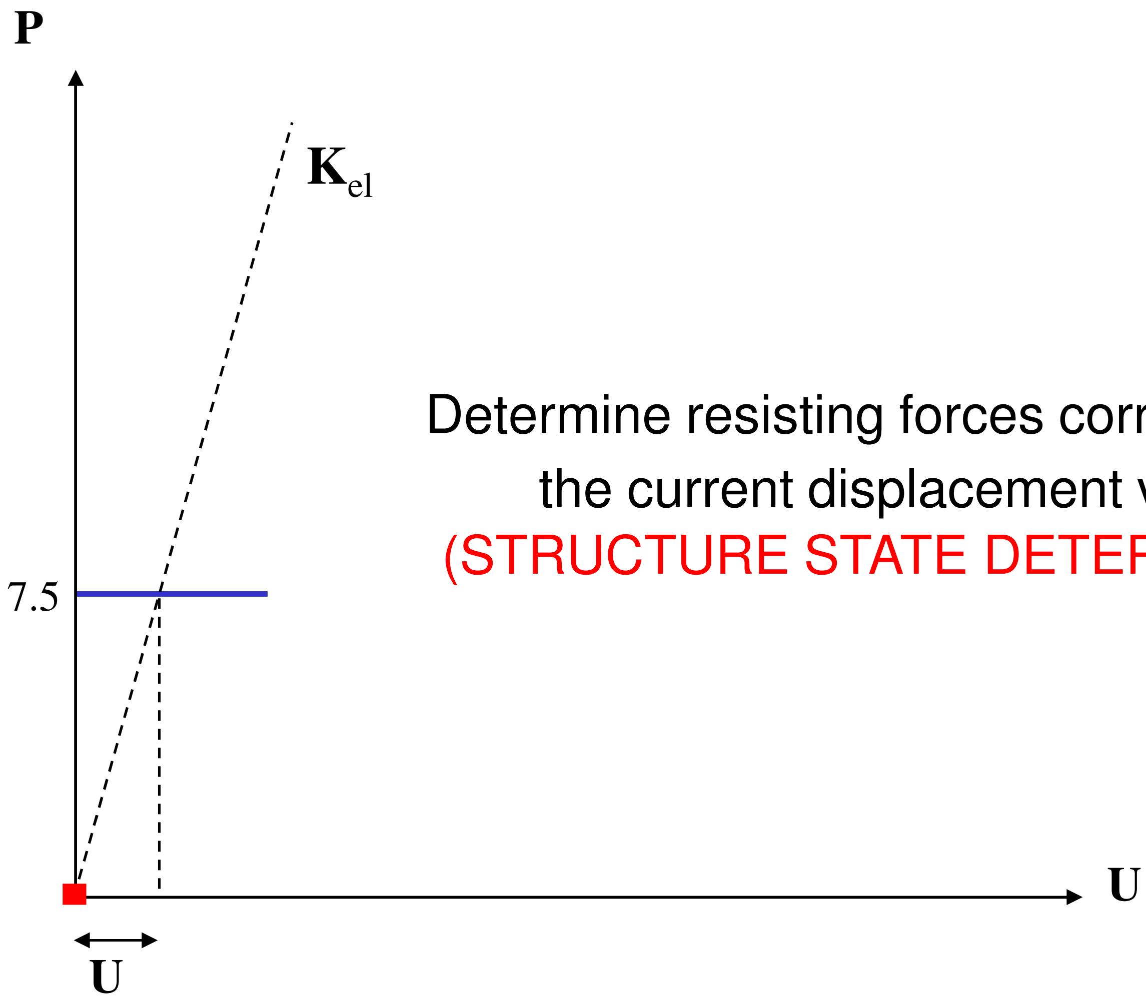
$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00045 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.00045$$

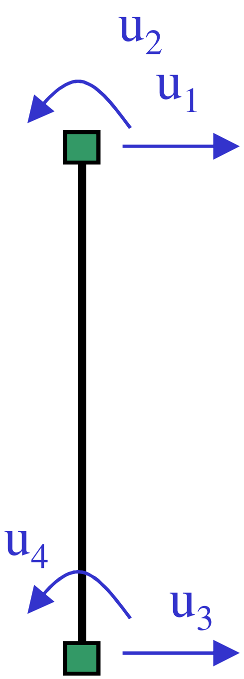


Example 1



Determine resisting forces corresponding to the current displacement vector $\mathbf{U}$
(STRUCTURE STATE DETERMINATION)

Example 1



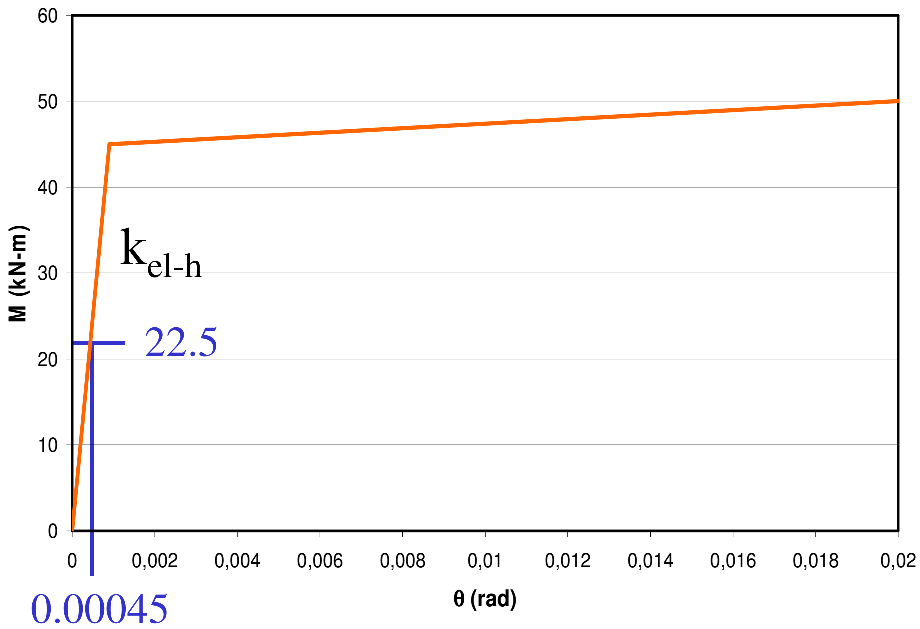
LOAD STEP 1: $\lambda_1 P = 7.5$ kN

$i=1$

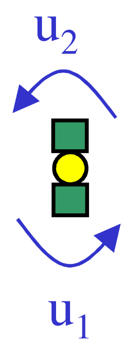
Elements' resisting forces
(ELEMENT STATE DETERMINATION)

1) Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

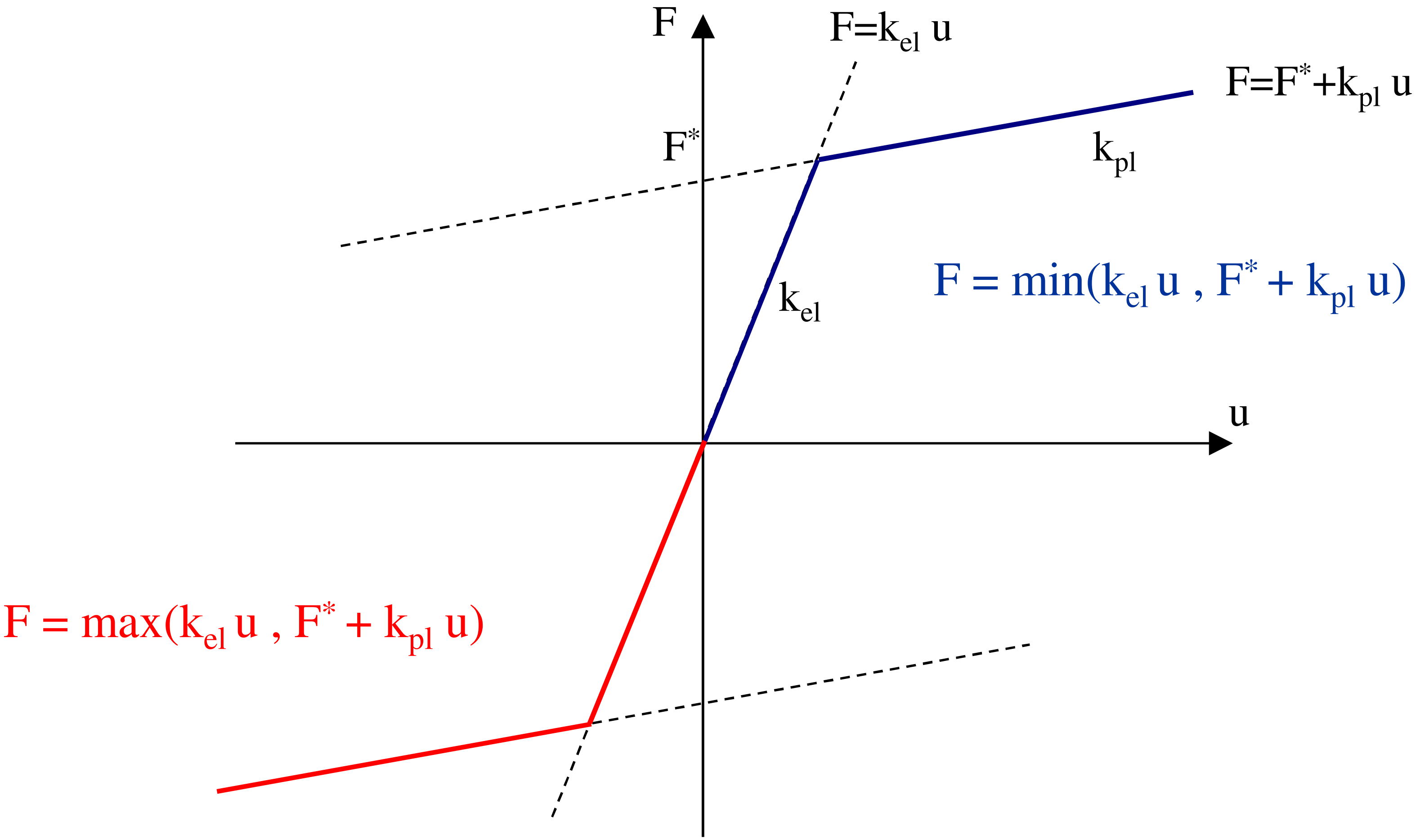
2) Plastic hinge $M_h = \min(k_{el\text{-}h}\,\theta_h\,,\, M^* + k_{pl\text{-}h}\,\theta_h) = -22.5$ kN-m



$k_h = k_{el\text{-}h}$



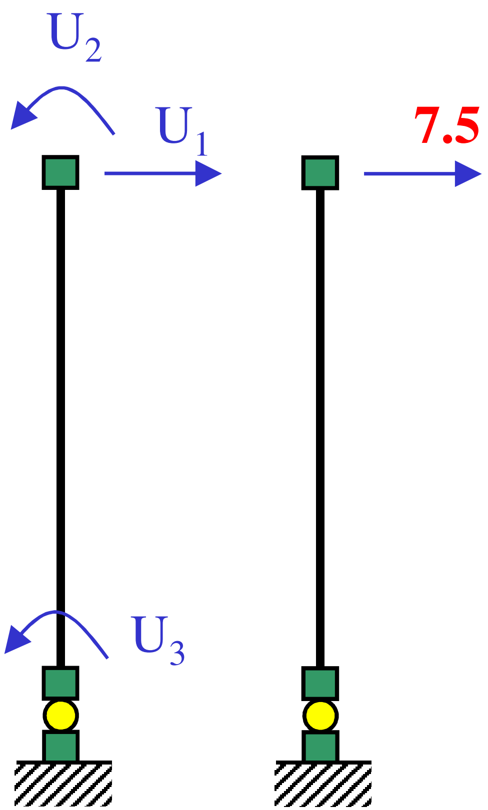
Example 1



Example 1

LOAD STEP 1: $\lambda_1 P = 7.5 \text{ kN}$

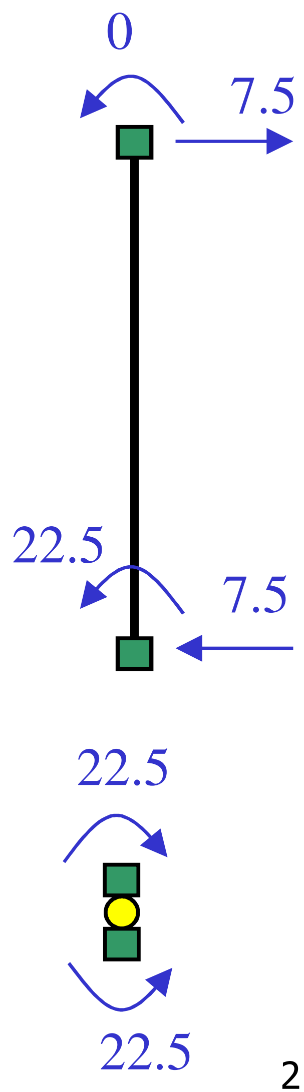
i=1



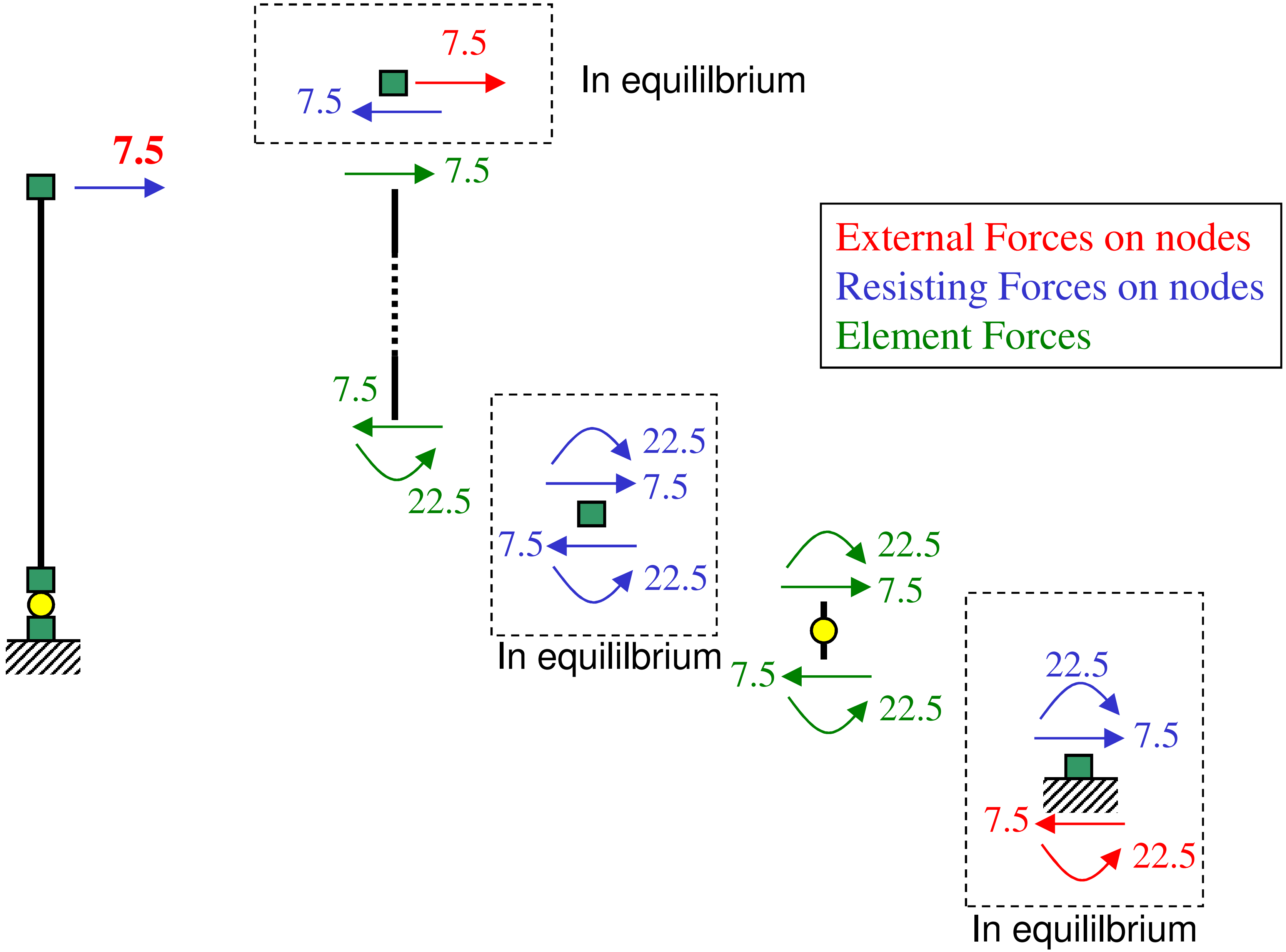
$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 22.5 \\ -22.5 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$

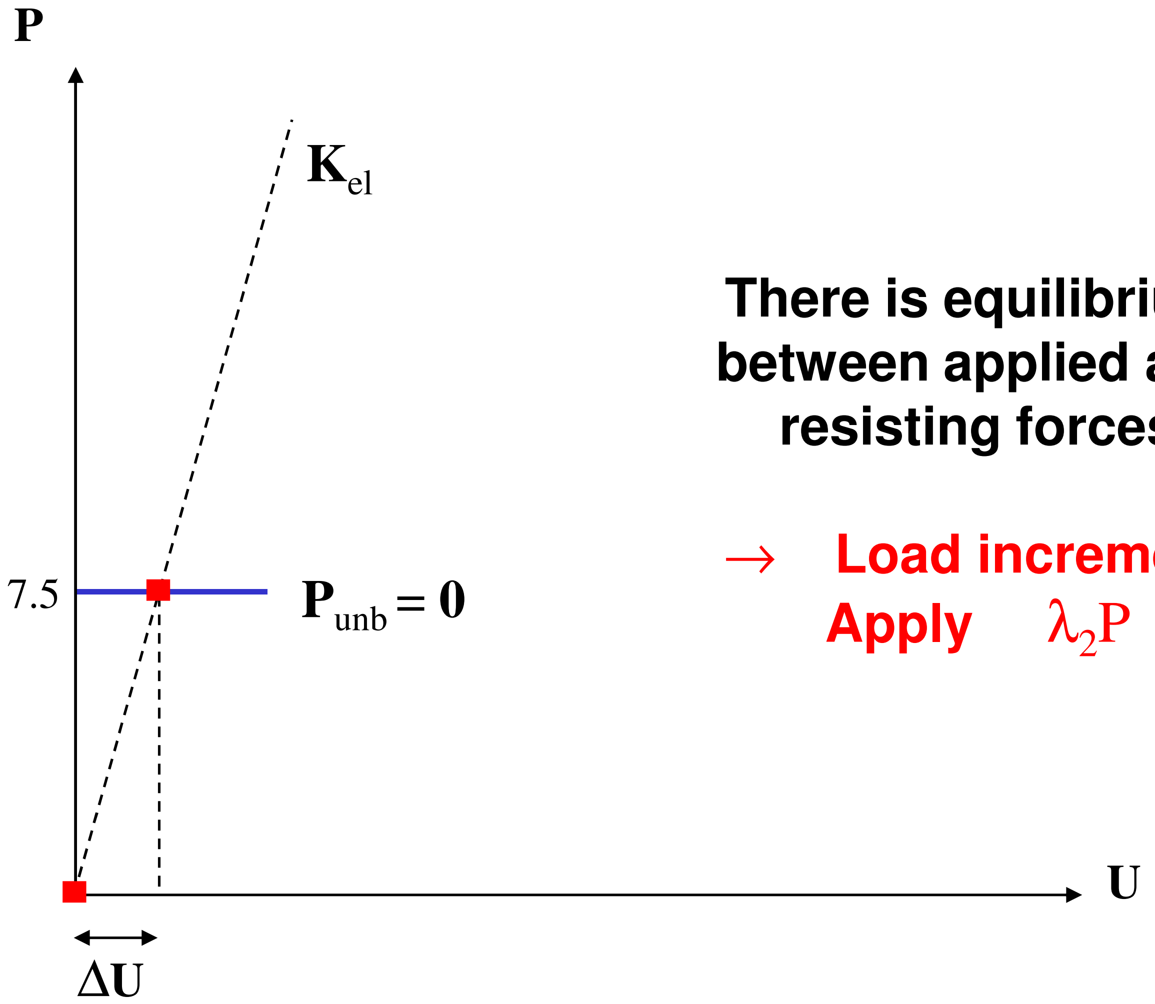
$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \mathbf{0}$$



Example 1


7.5
7.5
7.5
In equililbrium
7.5
7.5
22.5
22.5
7.5
7.5
22.5
In equililbrium
22.5
7.5
7.5
22.5
22.5
7.5
7.5
22.5
In equililbrium
External Forces on nodes
Resisting Forces on nodes
Element Forces

Example 1

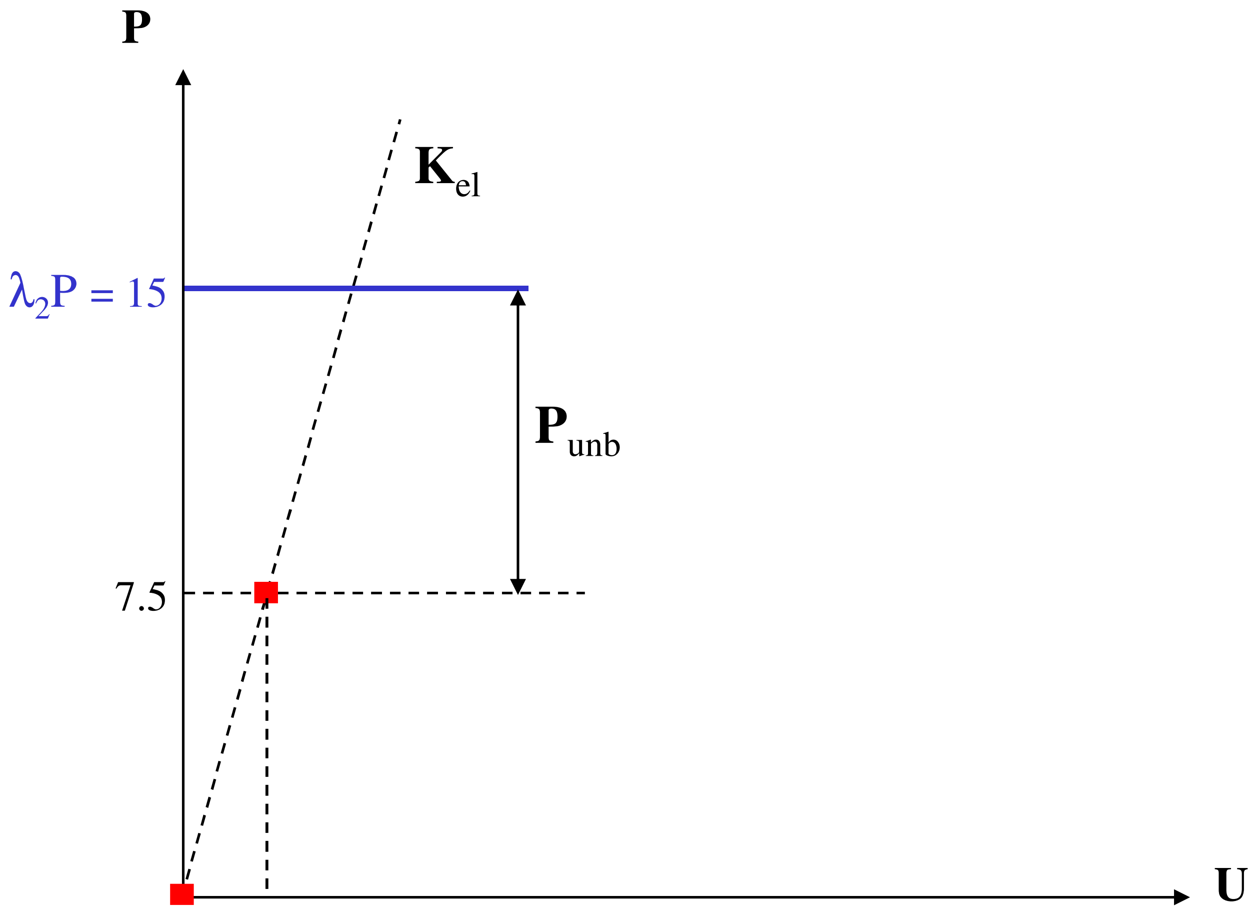


There is equilibrium between applied and resisting forces

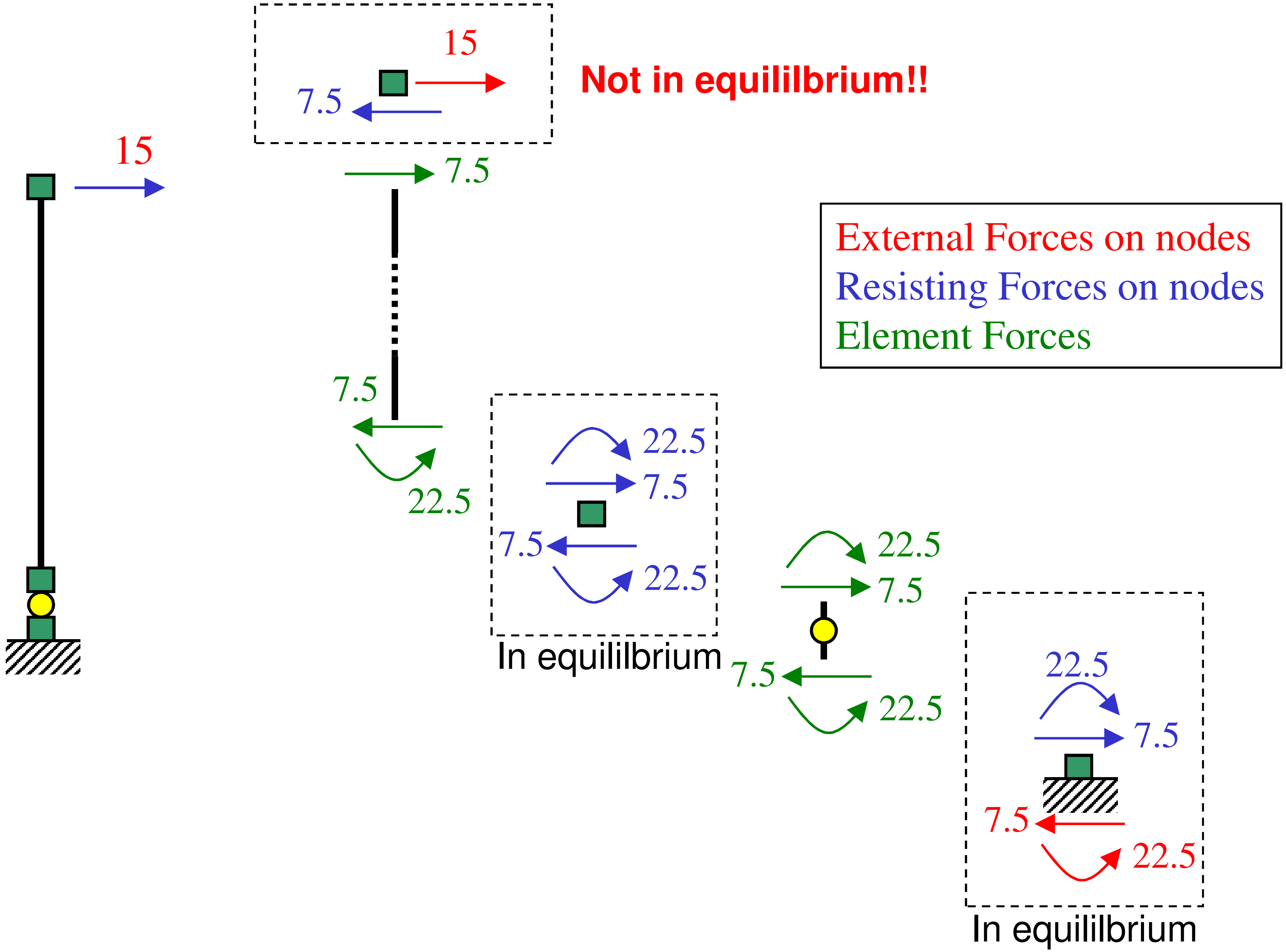
→ Load increment

Apply $\lambda_2 P$

Example 1



Example 1



15
15
7.5
Not in equililbrium!!
7.5
External Forces on nodes
Resisting Forces on nodes
Element Forces
7.5
22.5
22.5
7.5
7.5
22.5
In equililbrium
22.5
7.5
7.5
22.5
22.5
7.5
7.5
22.5
In equililbrium

Example 1

LOAD STEP 2: $\lambda_2 P = 15\ \text{kN}$

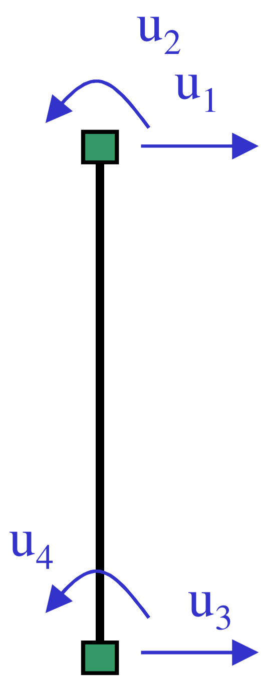
***i*=1**

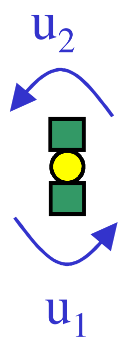
$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{P} = \begin{Bmatrix} 15 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$

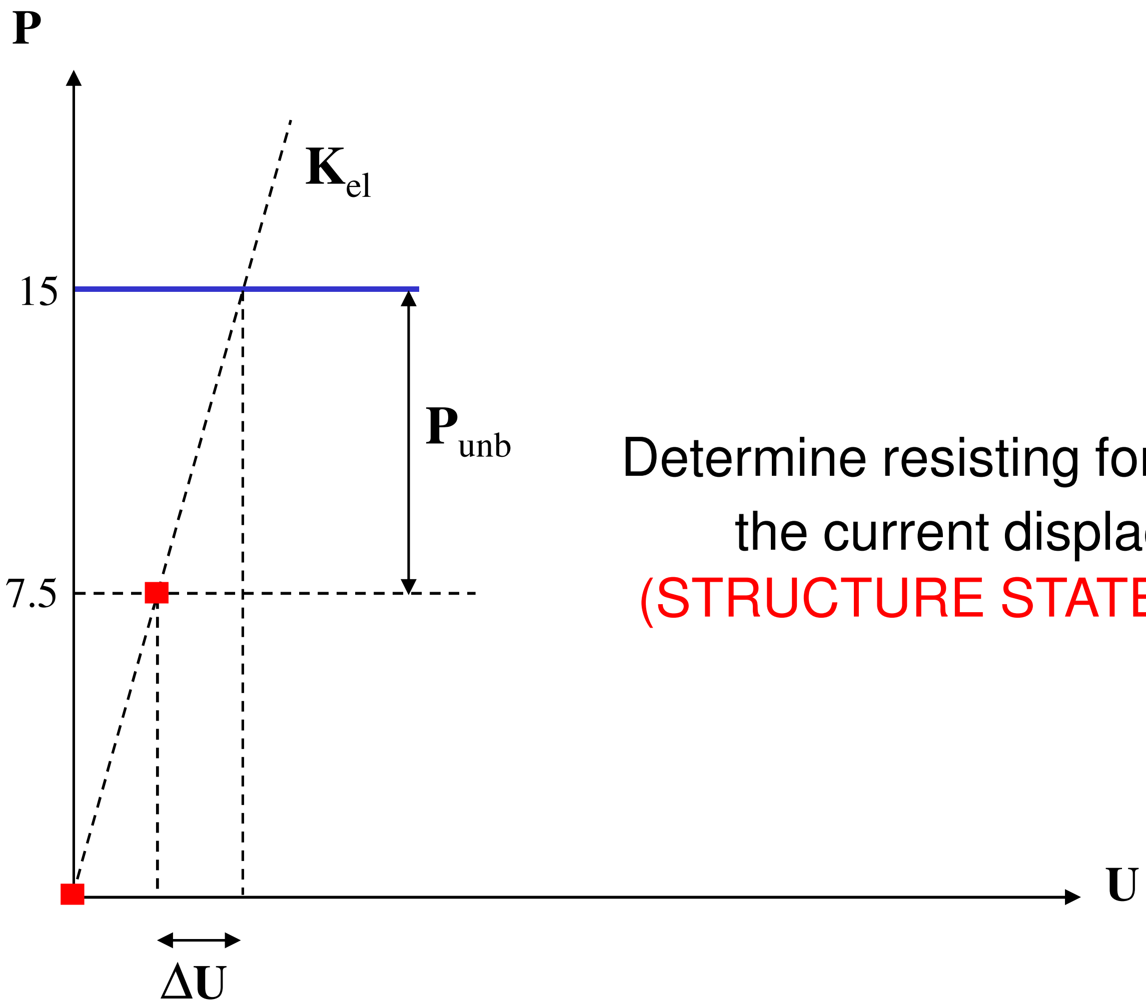
$$\mathbf{K} = \mathbf{K}_{el}$$

$$\Delta\mathbf{U} = \mathbf{K}^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix}$$





Example 1



Determine resisting forces corresponding to the current displacement vector **U**
(STRUCTURE STATE DETERMINATION)

Example 1

LOAD STEP 2: $\lambda_2 P = 15\ \text{kN}$

i=1

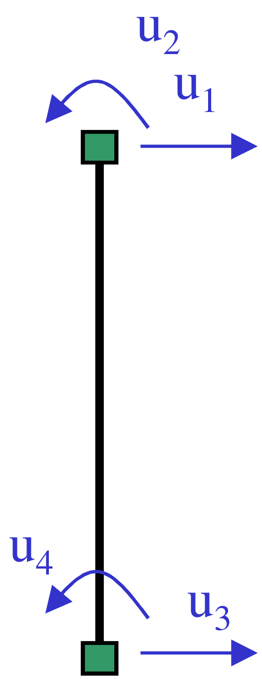
$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.0162 \\ -0.00765 \\ -0.0009 \end{Bmatrix}$$

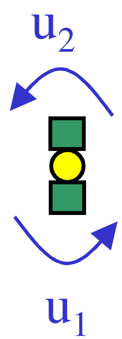
$$\mathbf{U}_b = \begin{Bmatrix} 0.0162 \\ -0.00765 \\ 0 \\ -0.0009 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.0009 \end{Bmatrix}$$

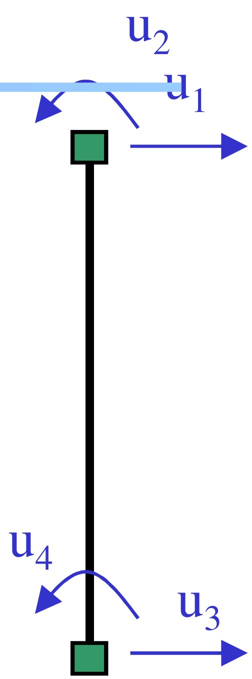
$$\Downarrow$$

$$\theta_h = -0.0009$$





Example 1



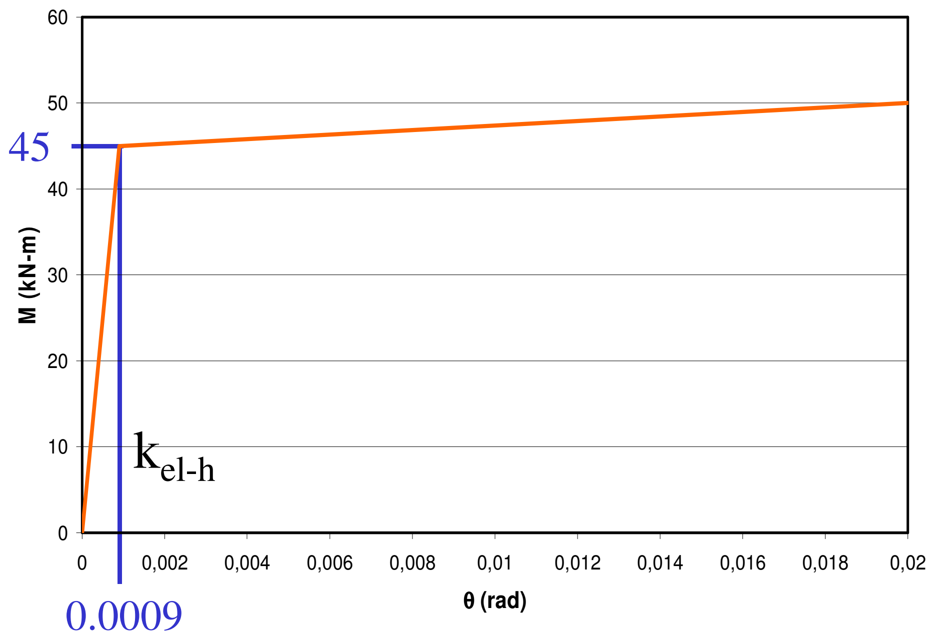
LOAD STEP 2: $\lambda_2 P = 15\ kN$

i=1

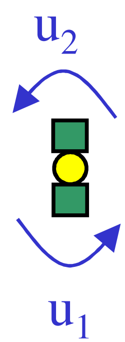
Elements' resisting forces
(ELEMENT STATE DETERMINATION)

1) Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

2) Plastic hinge $M_h = \min(k_{el\text{-}h}\ \theta_h\ ,\ M^* + k_{pl\text{-}h}\ \theta_h) = -45\ kN\text{-}m$



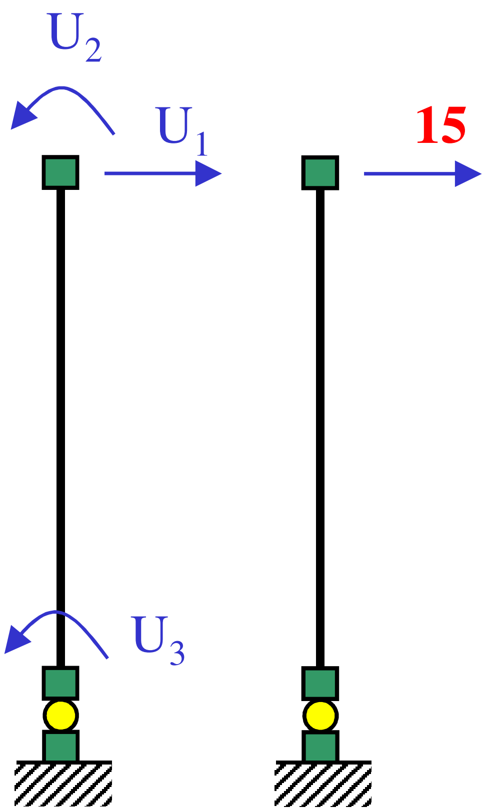
$k_h = k_{el\text{-}h}$



Example 1

LOAD STEP 2: $\lambda_2 P = 15$ kN

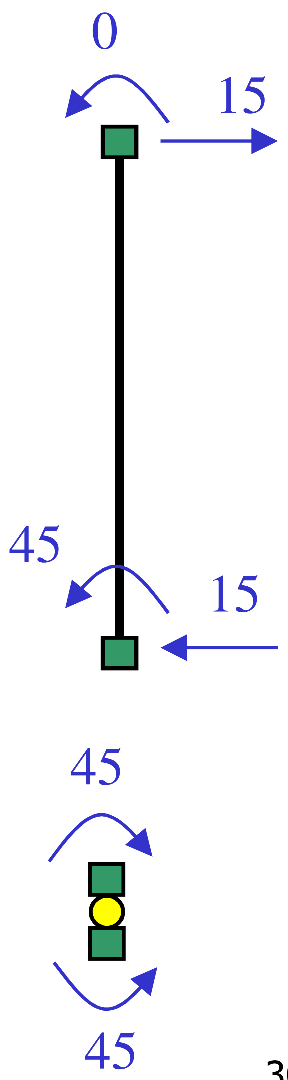
i=1



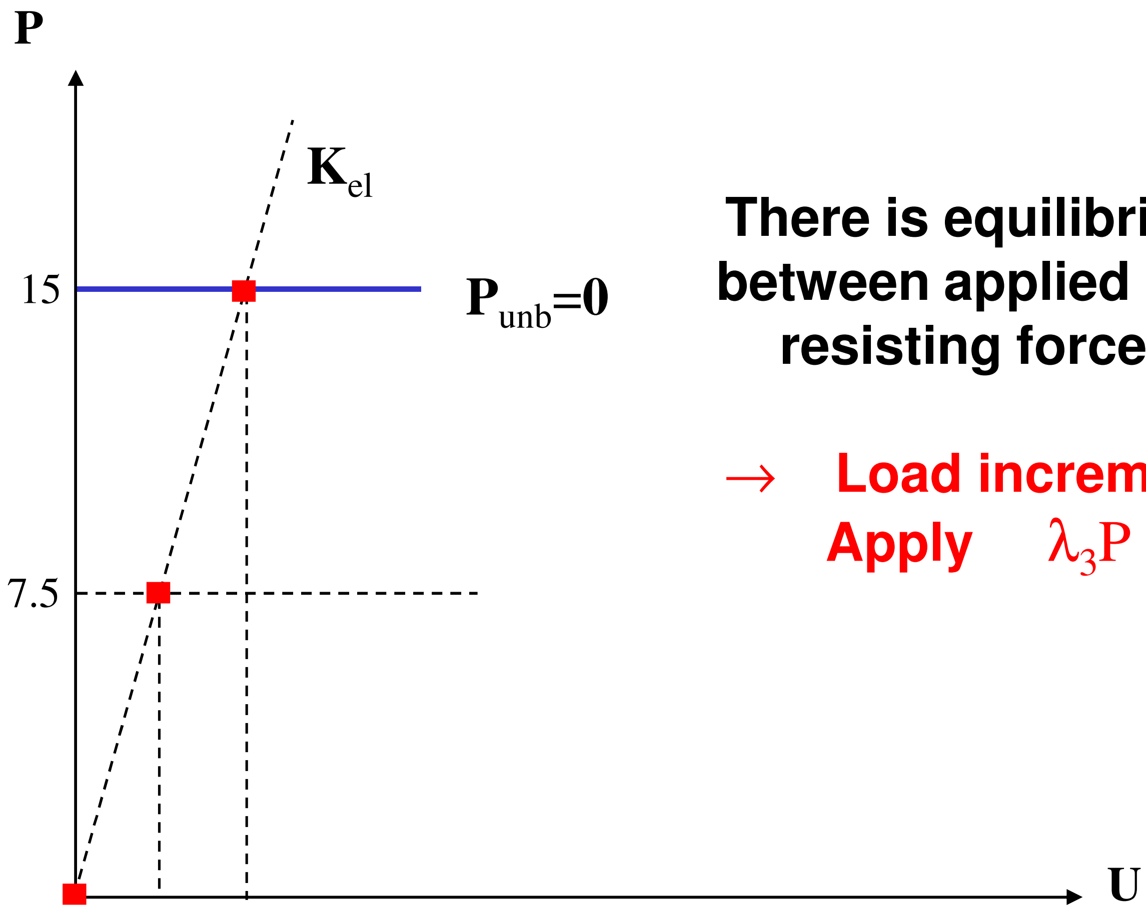
$$\mathbf{P}_b = \begin{Bmatrix} 15 \\ 0 \\ -15 \\ 45 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 45 \\ -45 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 15 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 15 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 15 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



Example 1



There is equilibrium between applied and resisting forces

→ **Load increment**
Apply $\lambda_3 P$

Example 1

LOAD STEP 3: $\lambda_3 P = 17\ \text{kN}$

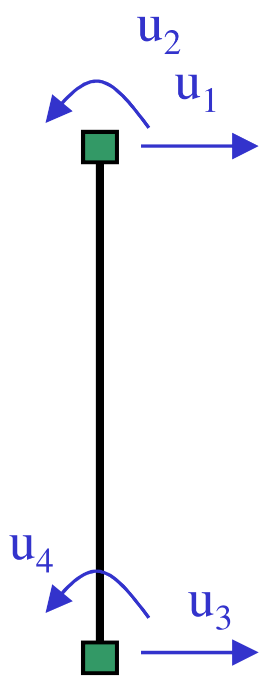
***i*=1**

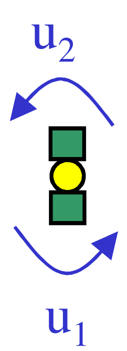
$$\mathbf{P}_R = \begin{Bmatrix} 15 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{P} = \begin{Bmatrix} 17 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 2 \\ 0 \\ 0 \end{Bmatrix}$$

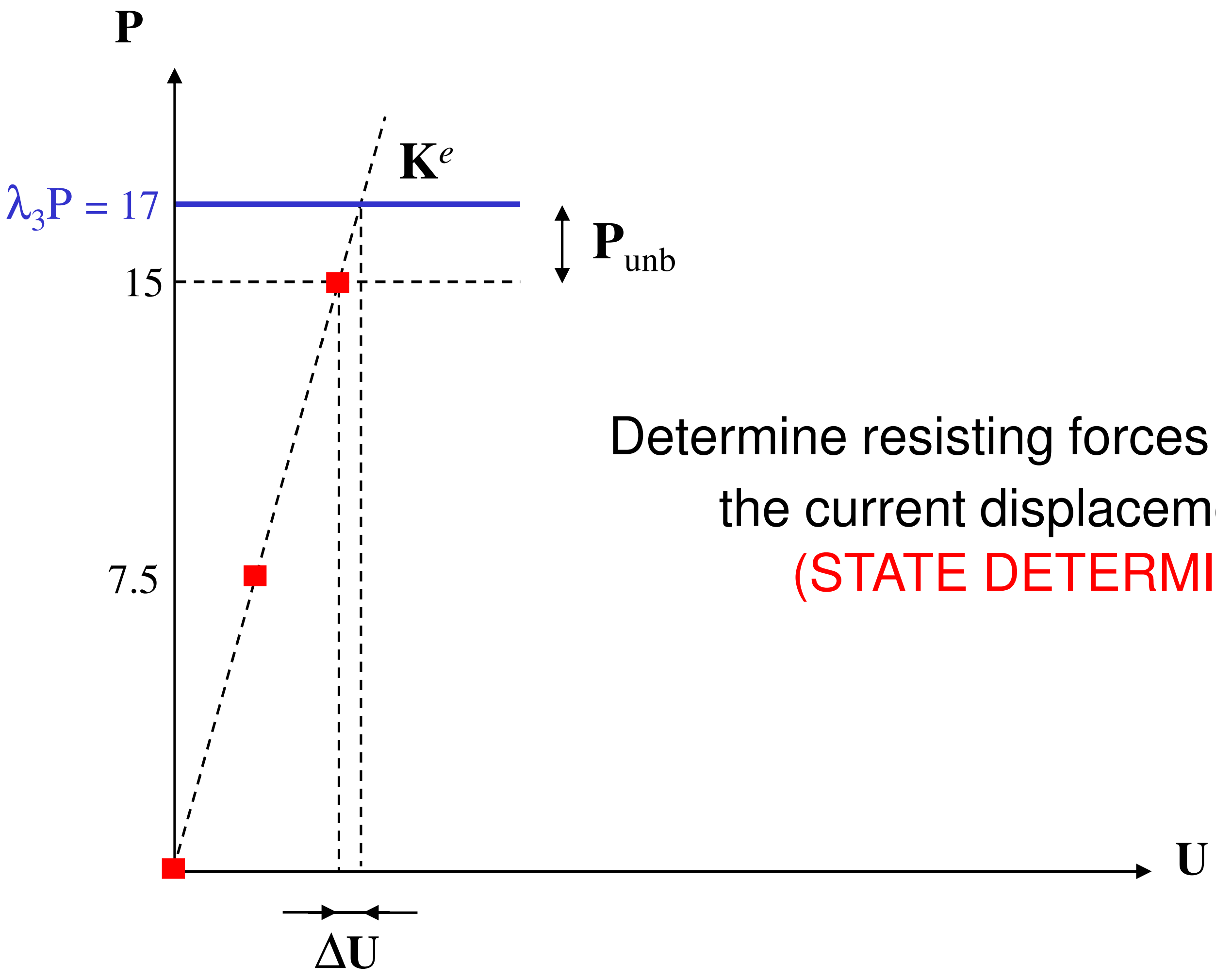
$$\mathbf{K} = \mathbf{K}_{el}$$

$$\Delta\mathbf{U} = \mathbf{K}^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.00216 \\ -0.001 \\ -0.00012 \end{Bmatrix}$$





Example 1



Determine resisting forces corresponding to the current displacement vector **U**

(STATE DETERMINATION)

Example 1

LOAD STEP 3: $\lambda_3 P = 17\ \text{kN}$

***i*=1**

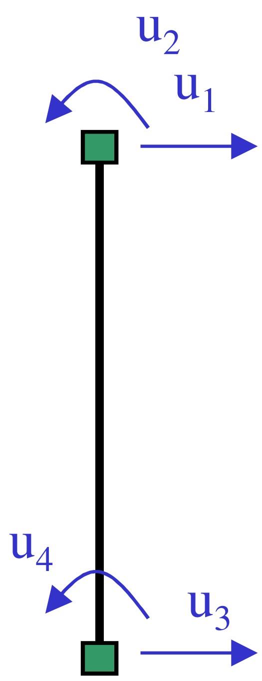
$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.01836 \\ -0.00867 \\ -0.00102 \end{Bmatrix}$$

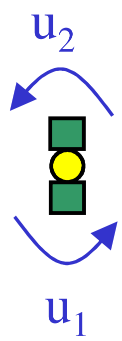
$$\mathbf{U}_b = \begin{Bmatrix} 0.01836 \\ -0.00867 \\ 0 \\ -0.00102 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00102 \end{Bmatrix}$$

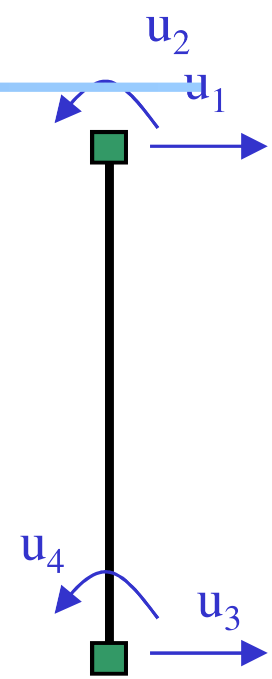
$$\Downarrow$$

$$\theta_h = -0.00102$$





Example 1



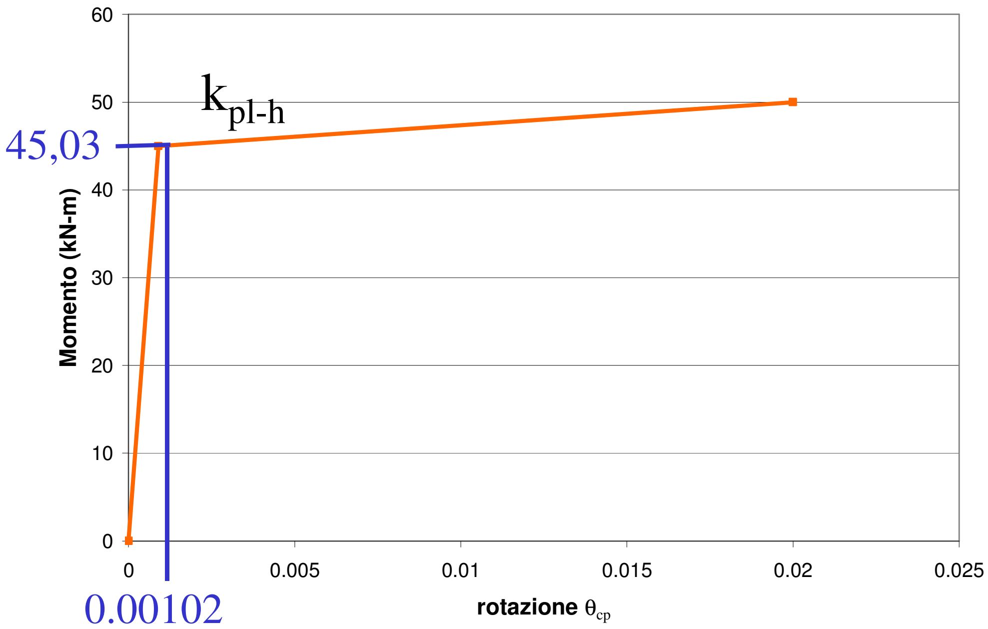
LOAD STEP 3: $\lambda_3 P = 17$ kN

i=1

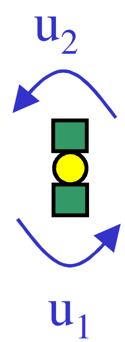
Elements' resisting forces (ELEMENT STATE DETERMINATION)

1) Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

2) Plastic hinge $M_h = \min(k_{el\text{-}h}\,\theta_h\,,\,M^* + k_{pl\text{-}h}\,\theta_h) = -45{,}03$ kN-m



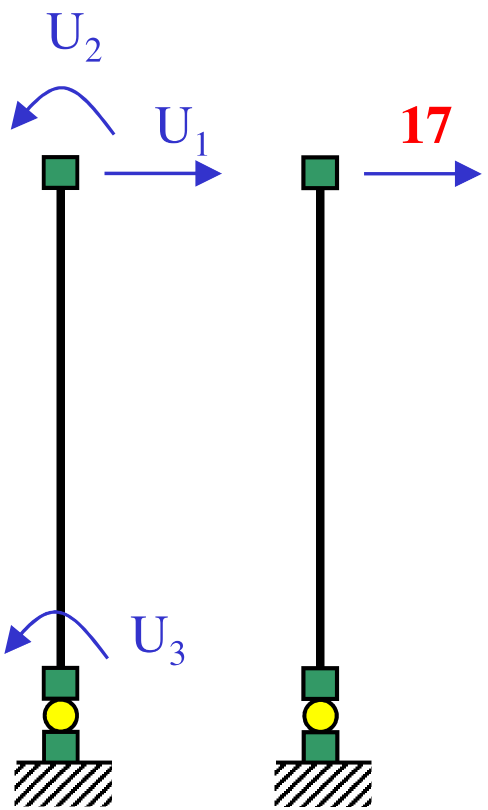
$k_h = k_{pl\text{-}h}$



Example 1

LOAD STEP 3: $\lambda_3 P = 17\ \text{kN}$

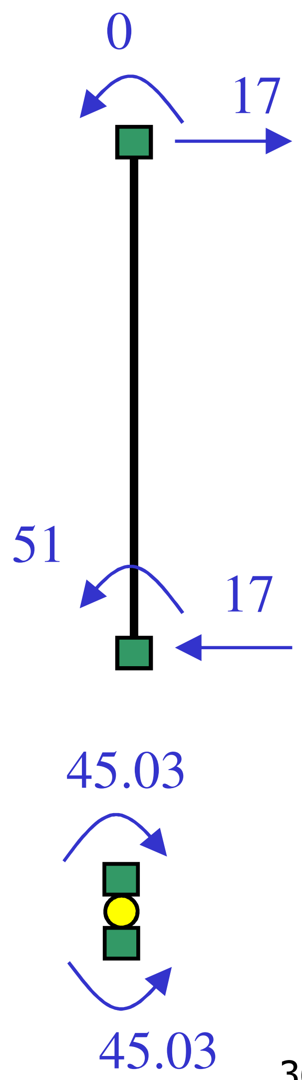
***i*=1**



$$\mathbf{P}_b = \begin{Bmatrix} 17 \\ 0 \\ -17 \\ 51 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 45.0312 \\ -45.0312 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 17 \\ 0 \\ 5.9688 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 17 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 17 \\ 0 \\ 5.9688 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -5.9688 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces

Example 1

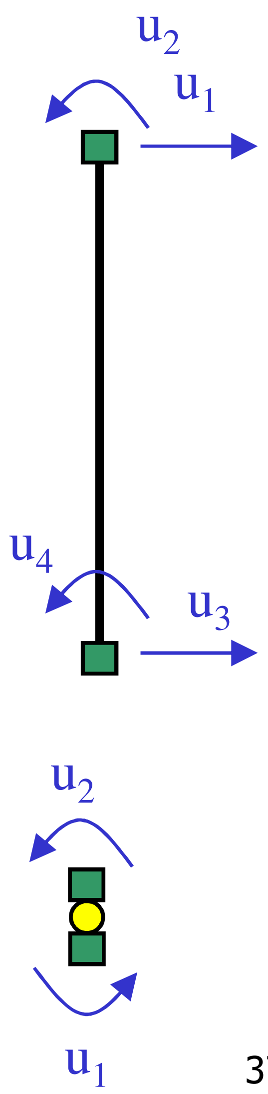
LOAD STEP 3: $\lambda_3 P = 17\ \text{kN}$

$i=2$

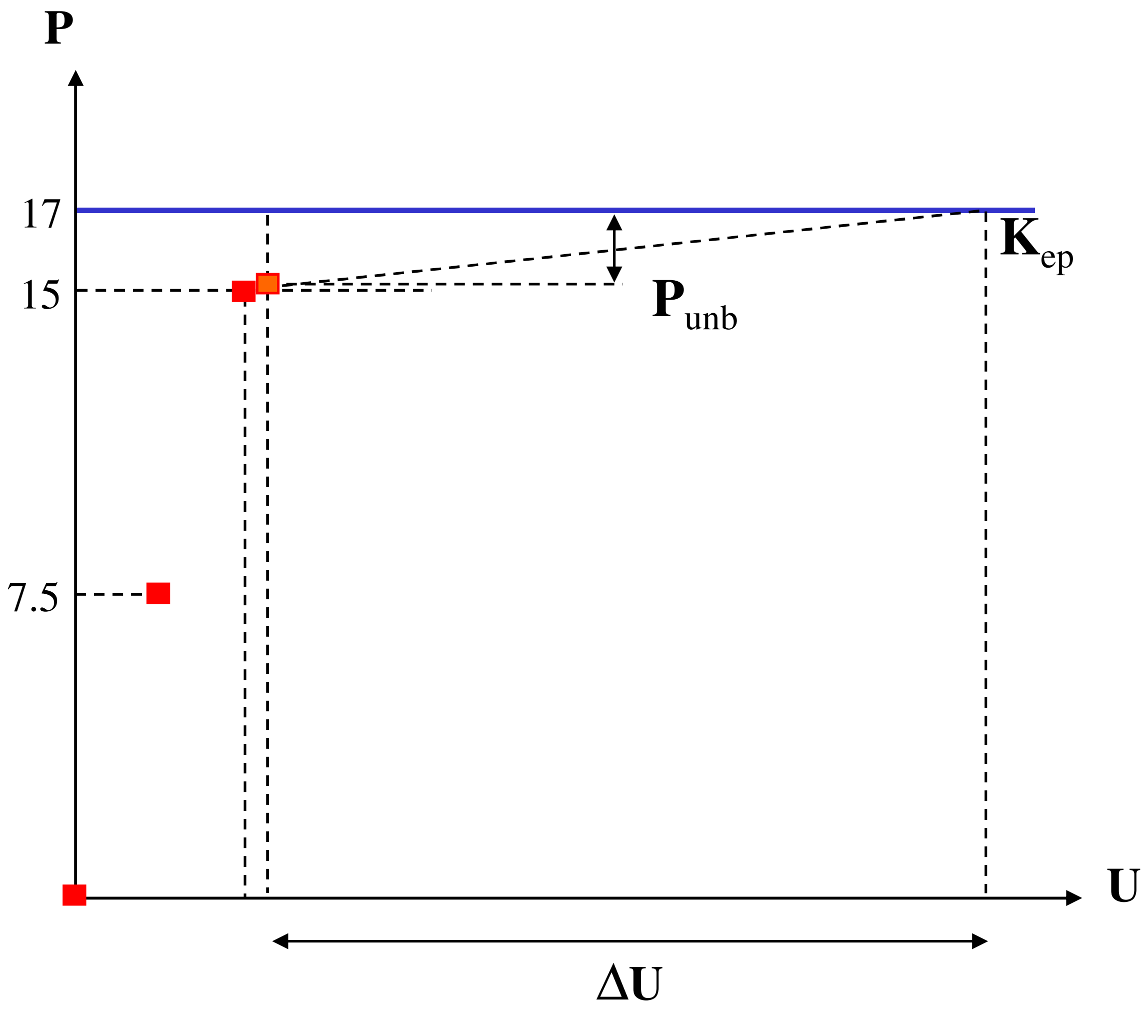
$$\mathbf{K} = \mathbf{K}_{pl}$$

$$\Delta\mathbf{U} = \mathbf{K}^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.06887 \\ -0.0229 \\ -0.0229 \end{Bmatrix}$$

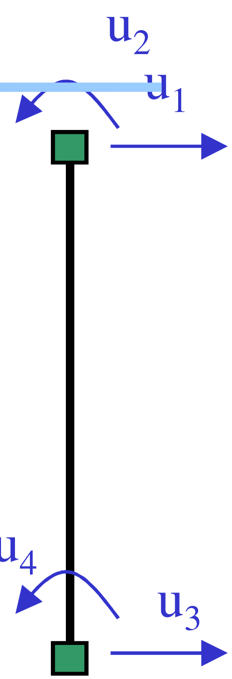
$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.08723 \\ -0.0316 \\ -0.0240 \end{Bmatrix}$$



Example 1



Example 1



LOAD STEP 3: $\lambda_3 P = 17$ kN

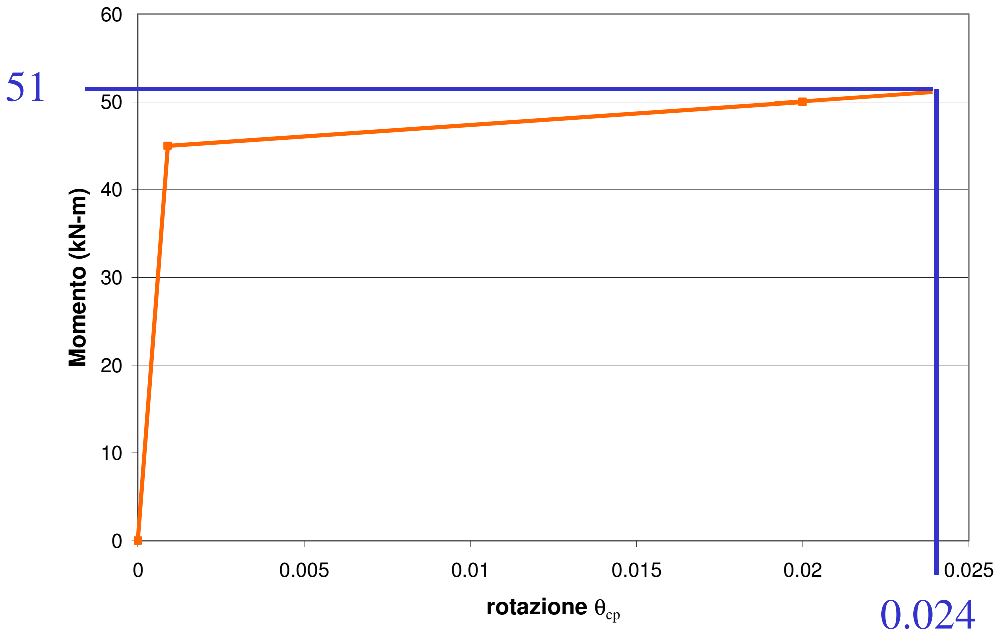
i=2

Elements' resisting forces
(ELEMENT STATE DETERMINATION)

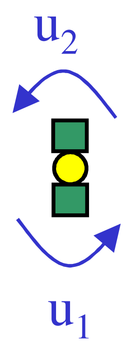
1) Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

2) Plastic hinge

$$M_h = \min(k_{el\text{-}h}\,\theta_h\ ,\ M^* + k_{pl\text{-}h}\,\theta_h) = -51 \text{ kN-m}$$



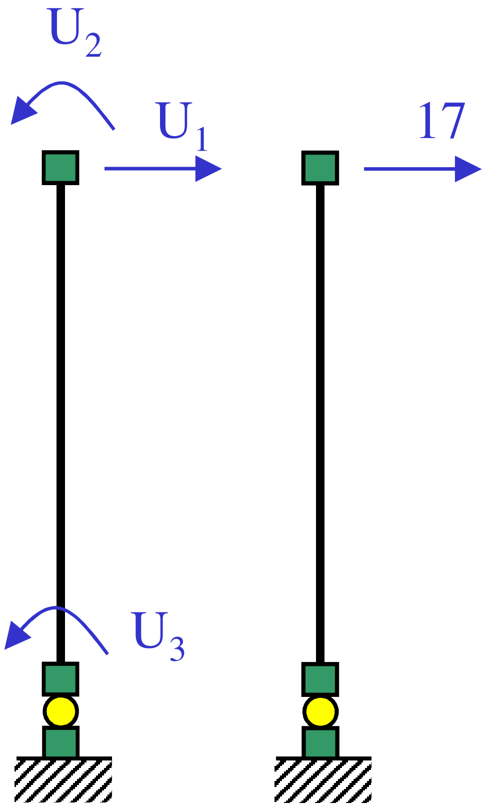
$K_h = k_{pl\text{-}h}$



Example 1

LOAD STEP 1: $\lambda_3 P = 17\ \text{kN}$

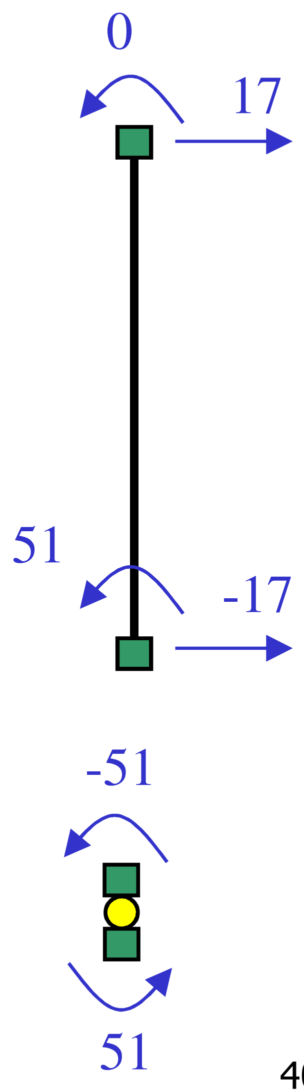
i=2



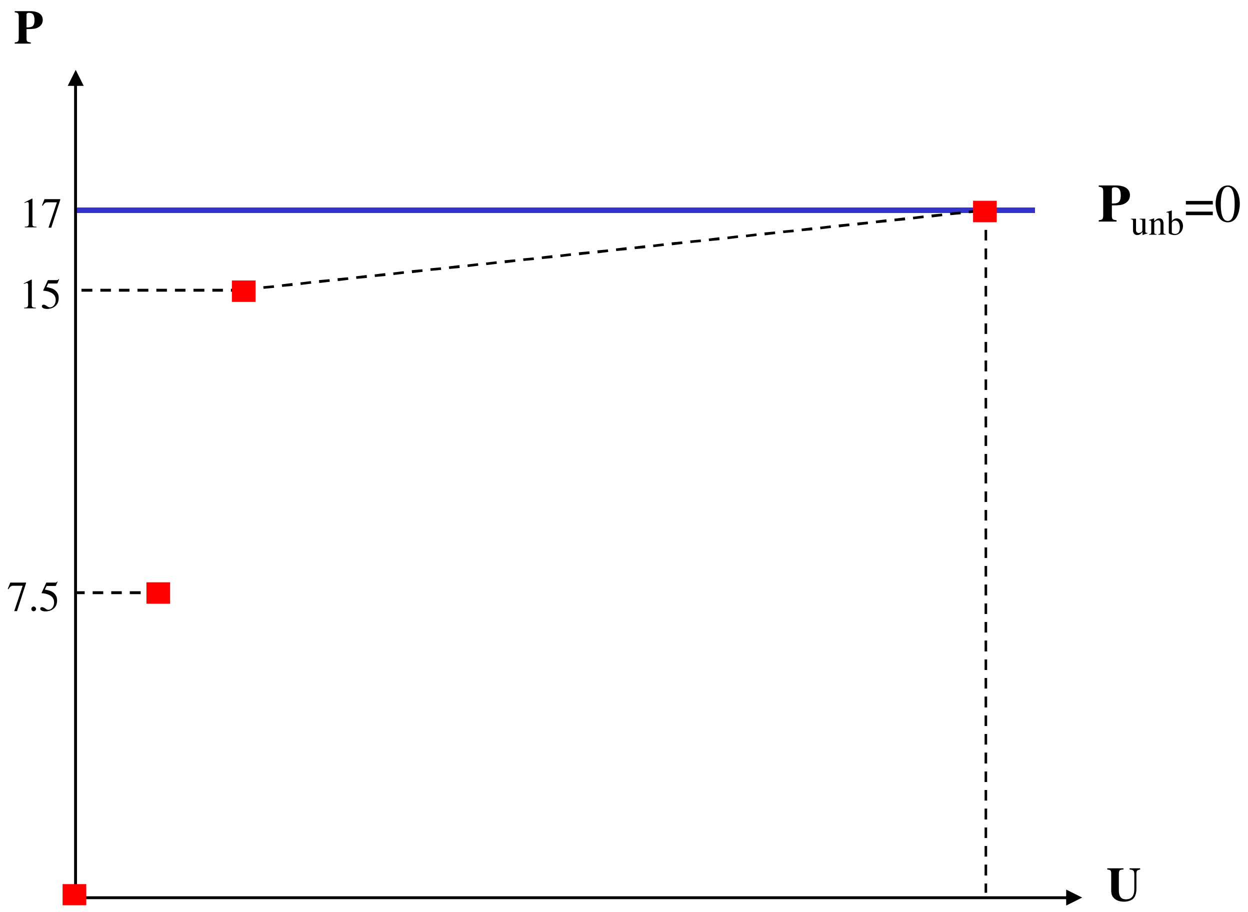
$$\mathbf{P}_b = \begin{Bmatrix} 17 \\ 0 \\ -17 \\ 51 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 51 \\ -51 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 17 \\ 0 \\ 0 \end{Bmatrix}$$

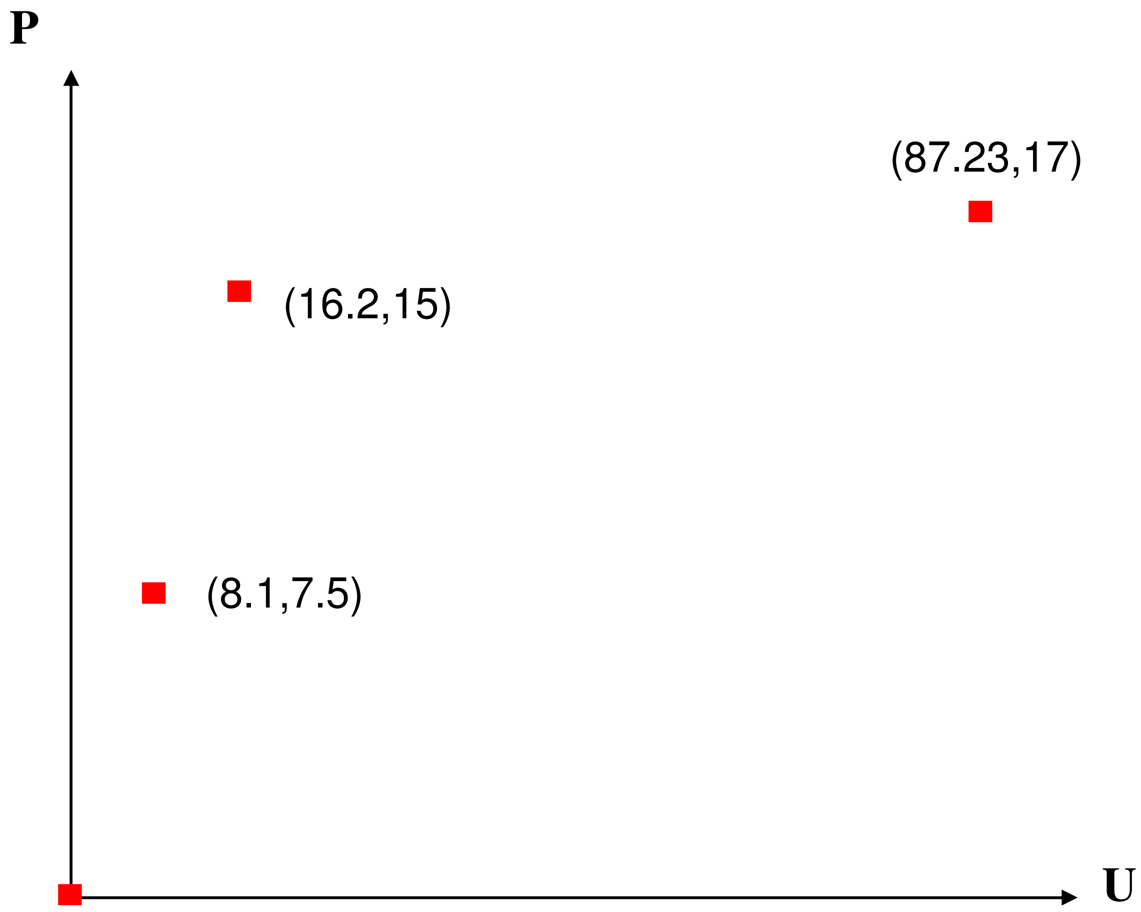
$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 17 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 17 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



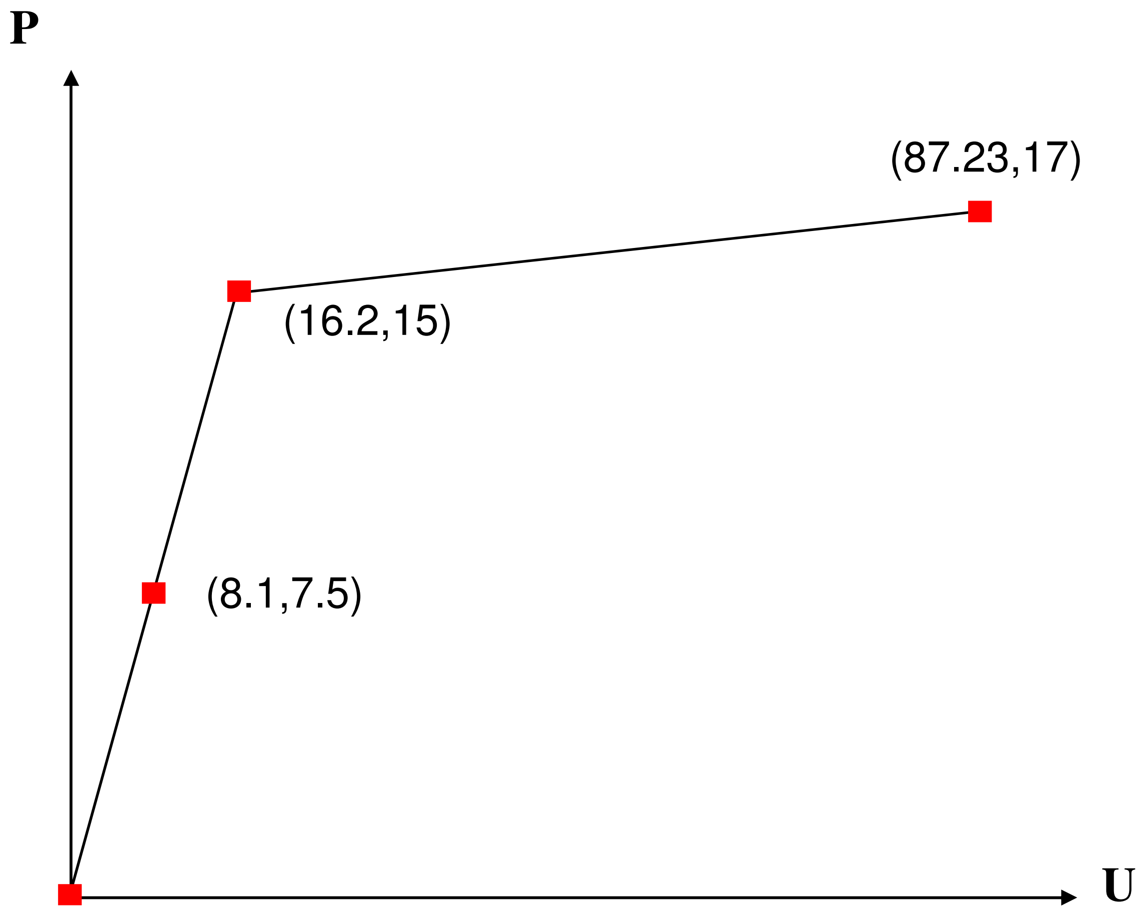
Example 1



Example 1

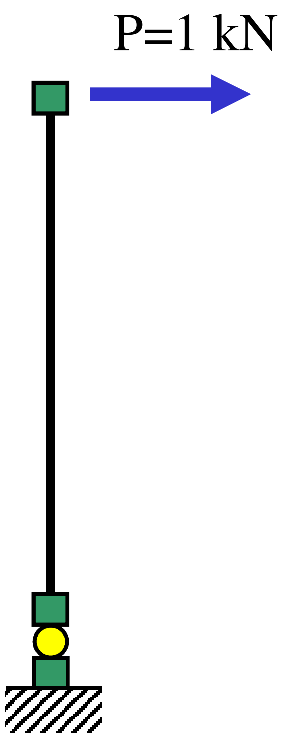


Example 1



Example 1

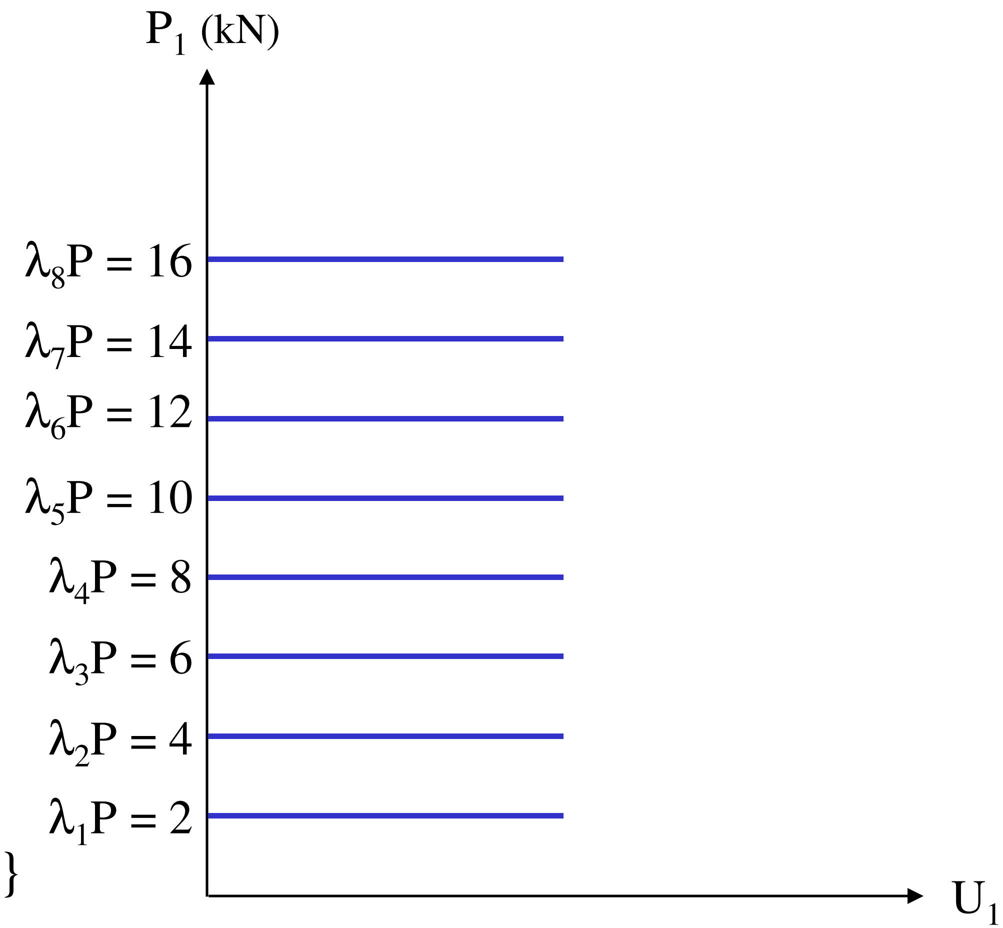
The load path is not known: A load history is the applied



Load history

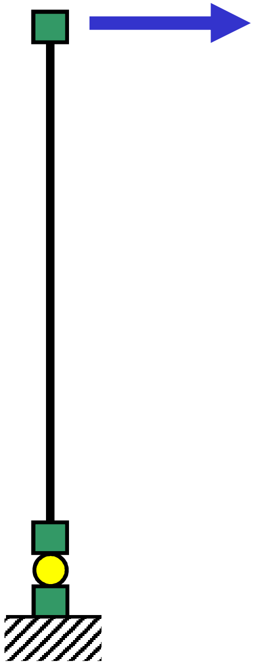
$P_1 = \lambda P$

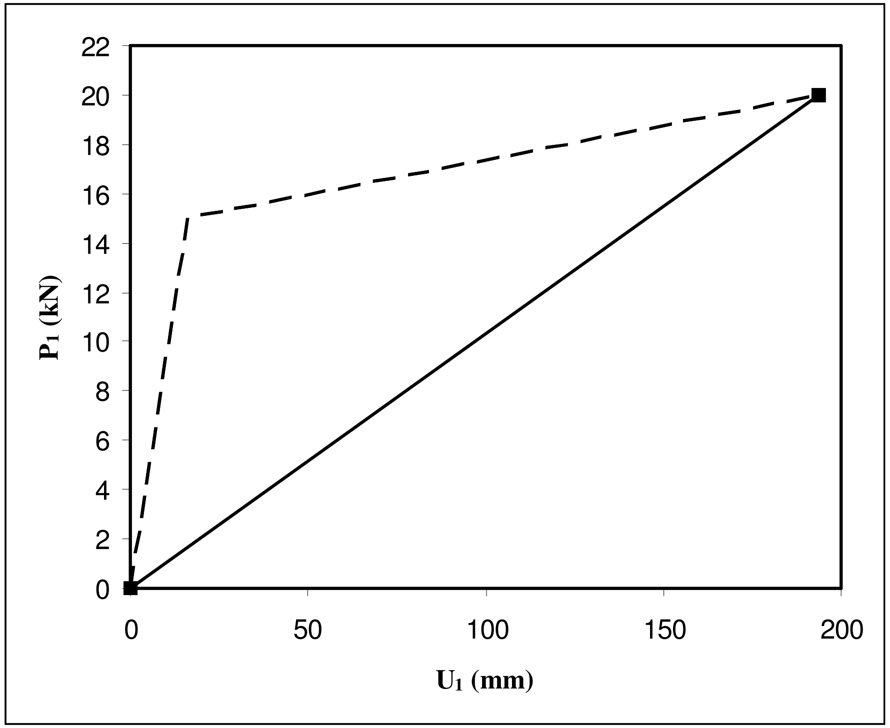
$\lambda = \{2, 4, 6, 8, 10, 12, 14, 16, \dots\}$



Example 1

P=1 kN



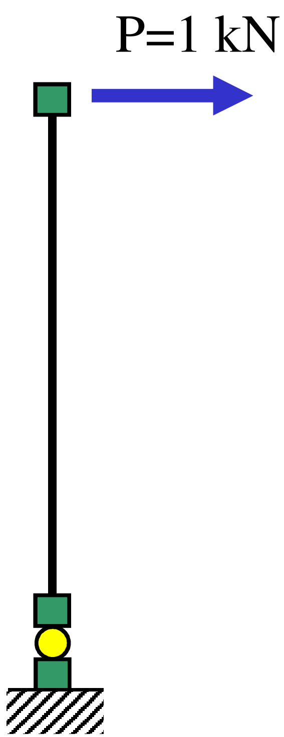


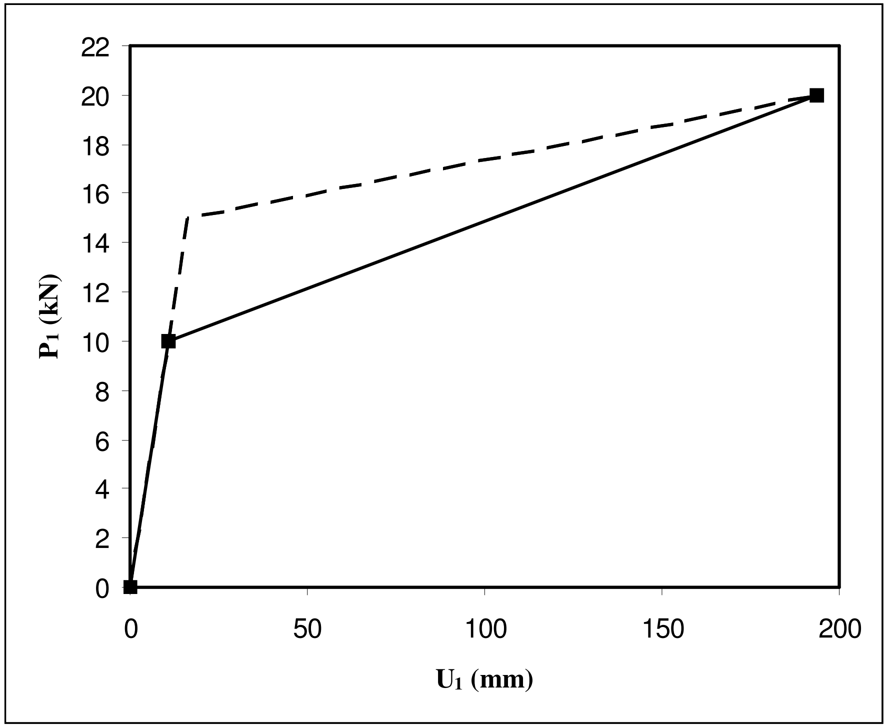
Load history

$P_1 = \lambda P$

$\lambda = \{0, 20\}$

Example 1



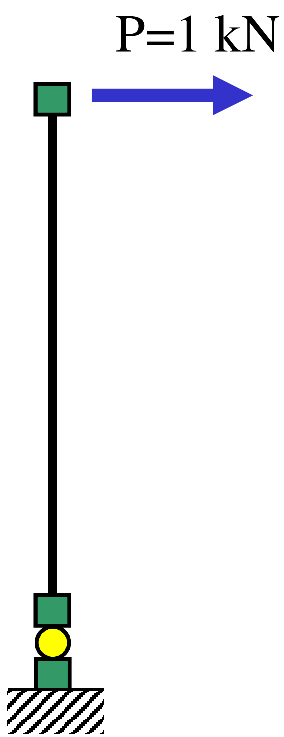


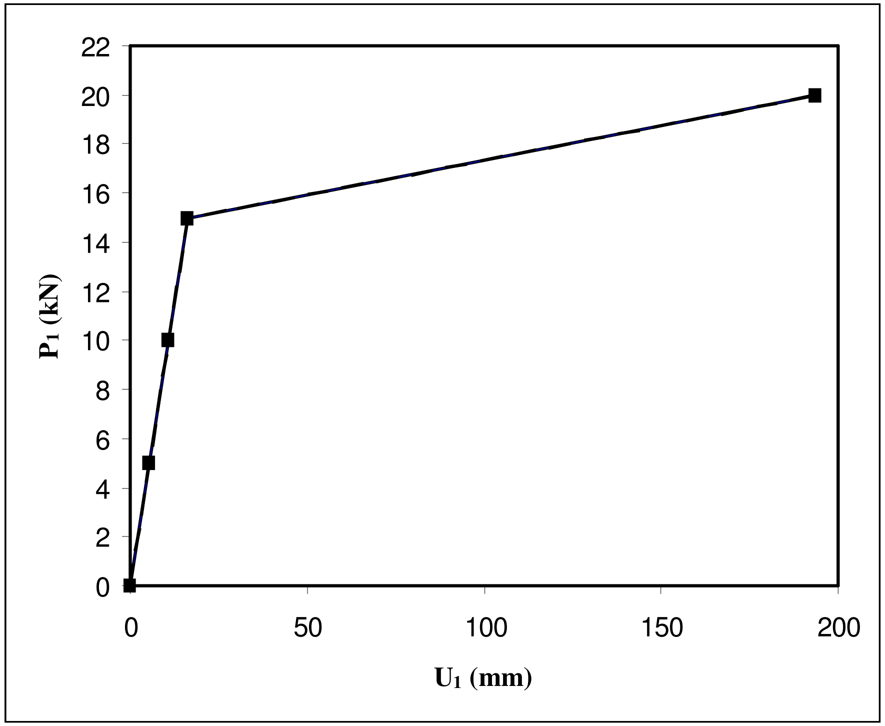
Load history

$P_1 = \lambda P$

$\lambda = \{0, 10, 20\}$

Example 1



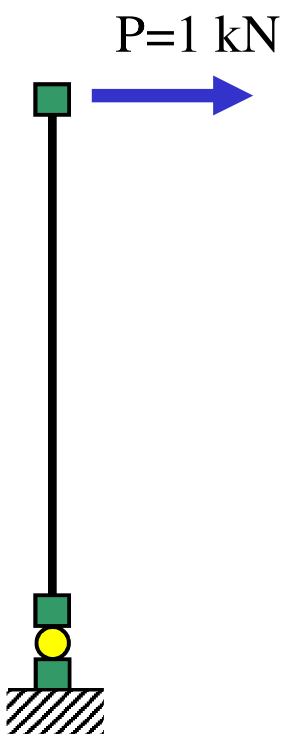


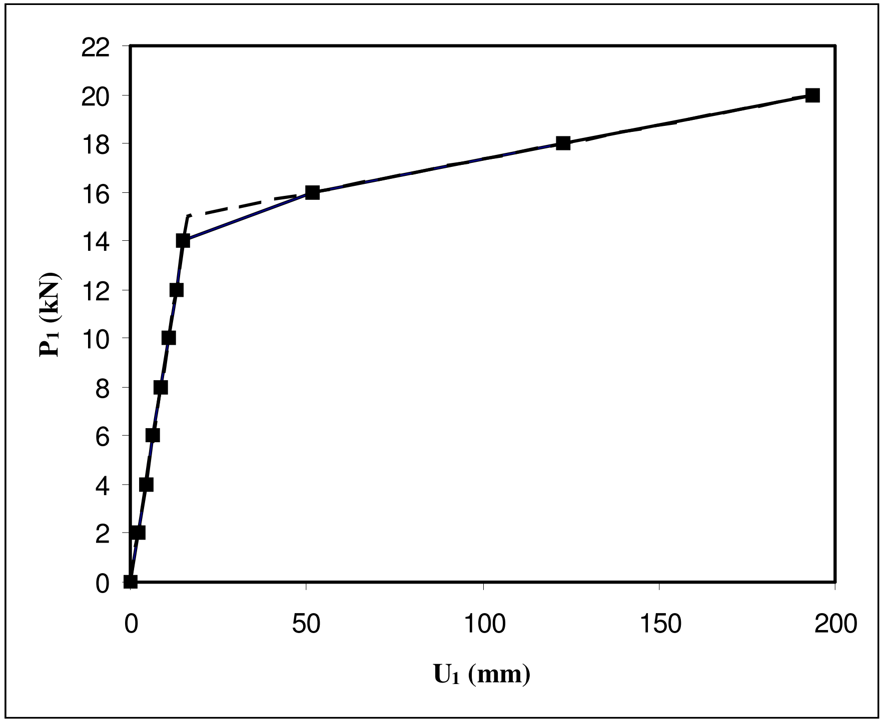
Load history

$P_1 = \lambda P$

$\lambda = \{0, 5, 10, 15, 20\}$

Example 1





Load history

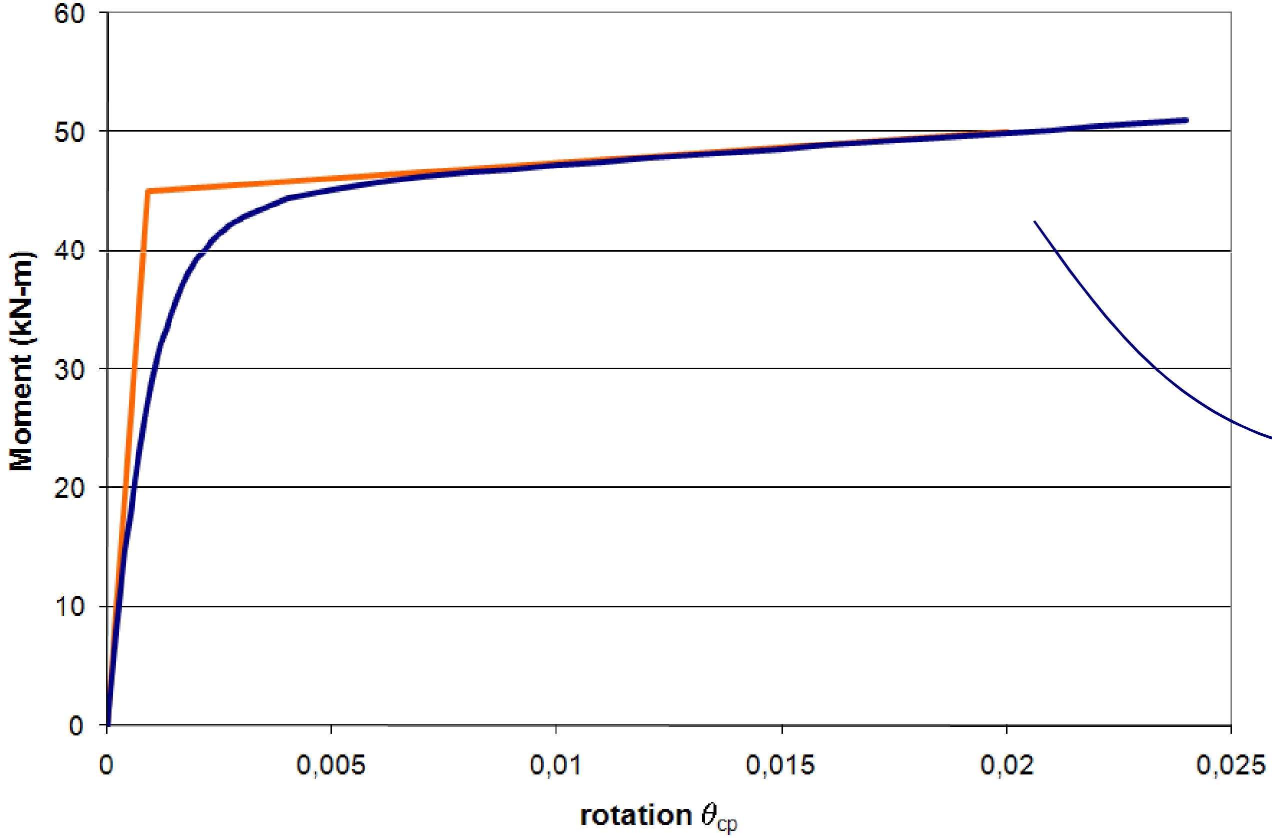
$P_1 = \lambda P$

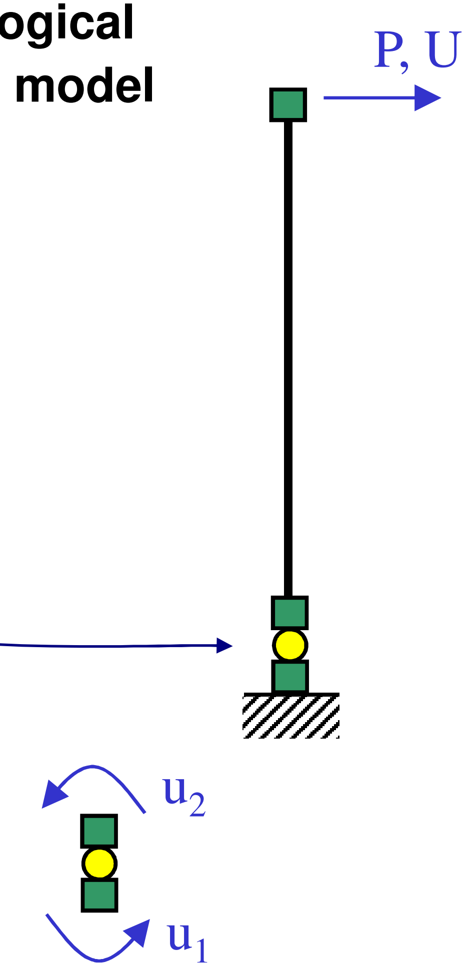
$\lambda = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

Example 2

PLASTIC HINGE

Phenomenological nonlinear M-θ model

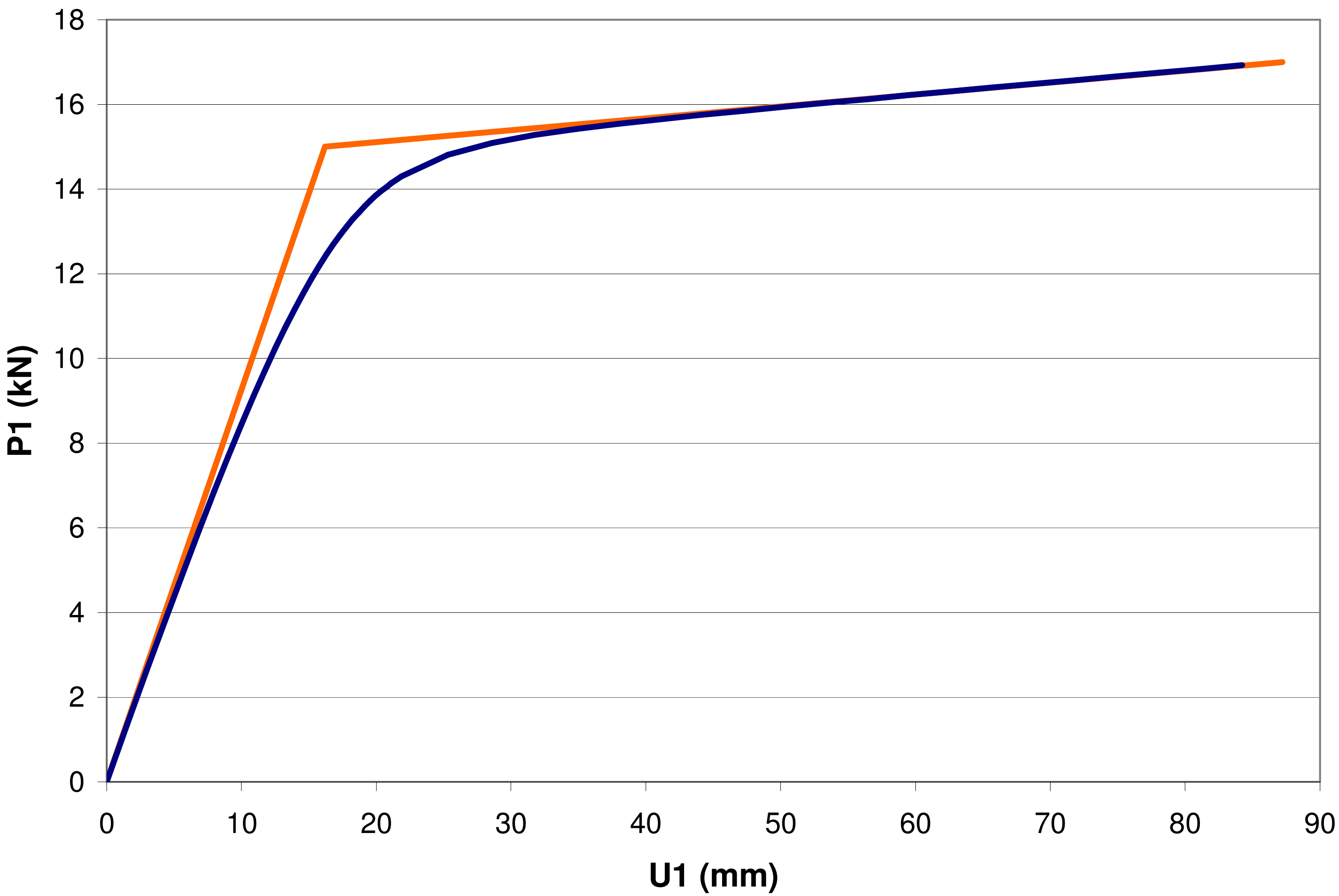


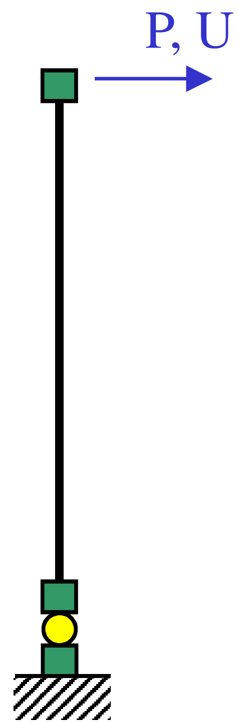


rotation = u_2-u_1

Example 2

SYSTEM RESPONSE
(closed form solution)





Example 2

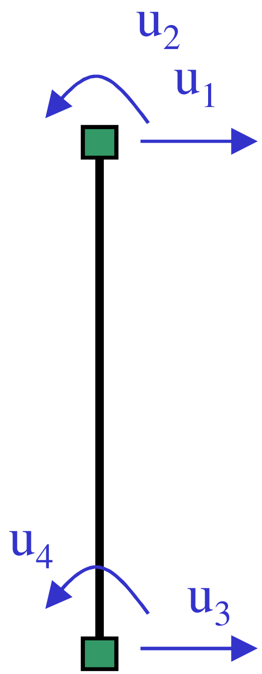
LOAD STEP 1: $\lambda_1 P = 7.5 \text{ kN}$

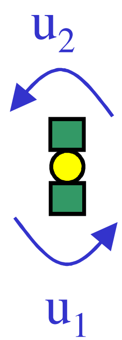
i=1

$$\mathbf{U} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{P}_b = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{P}_h = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{P}_R = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{P} = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$



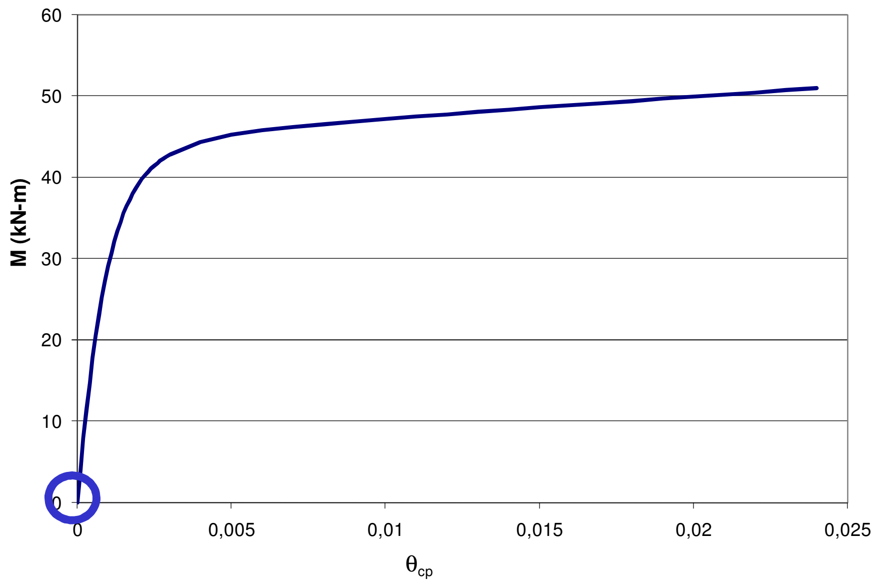


Example 2

LOAD STEP 1: $\lambda_1 P = 7.5$ kN

i=1

Initial stiffness



$EI_b = 10^4$ kN-m^2

$k_h = k_{el\text{-}h} = EI_{el\text{-}h}/L_{pl} = 5 \times 10^4$ kN-m

$L_b = 3$ m

$$\mathbf{K}_0 = \mathbf{K}_{el} = \begin{bmatrix} \frac{12EI_b}{L_b^3} & \frac{6EI_b}{L_b^2} & \frac{6EI_b}{L_b^2} \\ \frac{6EI_{el}}{L_b^2} & \frac{4EI_b}{L_b} & \frac{2EI_b}{L_b} \\ \frac{6EI_b}{L_b^2} & \frac{2EI_b}{L_b} & \frac{4EI_b}{L_b} + k_{el-h} \end{bmatrix}$$

Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

***i*=1**

$$\mathbf{\Delta U = K_0^{-1}\{\Delta P\}} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix}$$

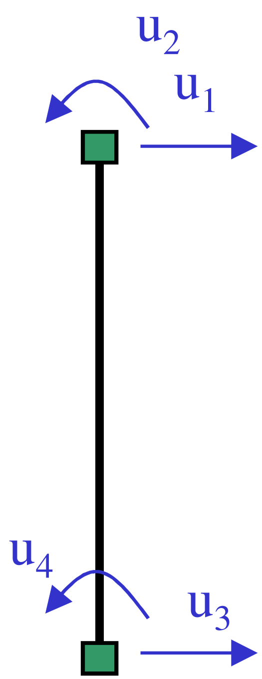
$$\mathbf{U = U + \Delta U} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix}$$

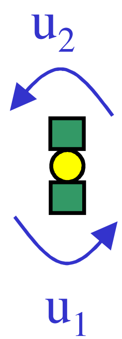
$$\mathbf{U}_b = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ 0 \\ -0.00045 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00045 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.00045$$





Example 2

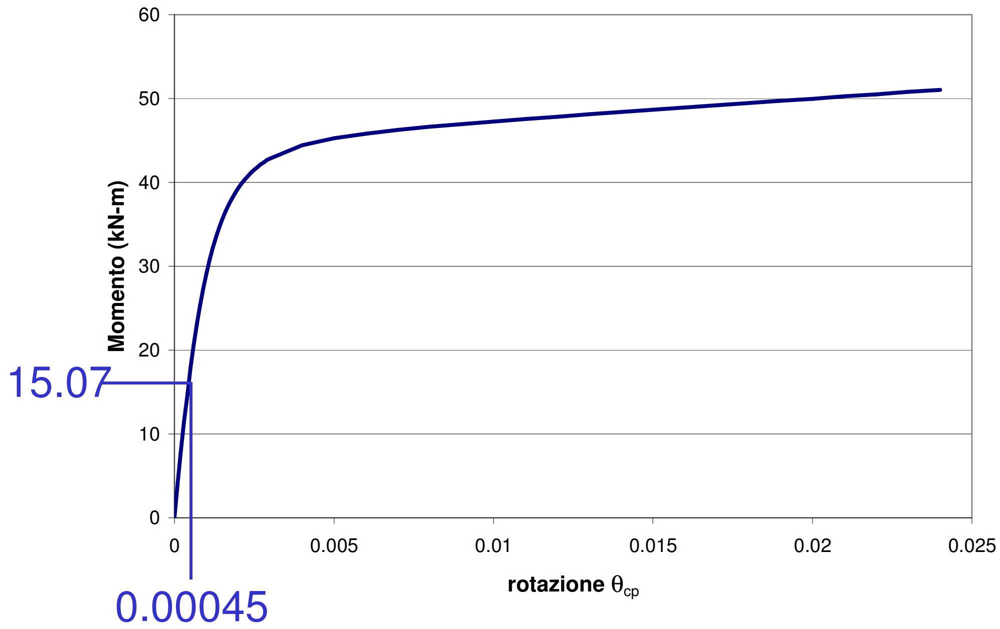
LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

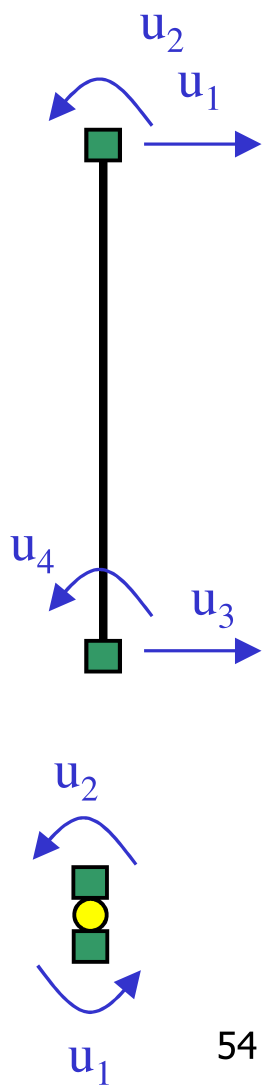
i=1

Elements' resisting forces

Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -15.07\ \text{kN-m}$

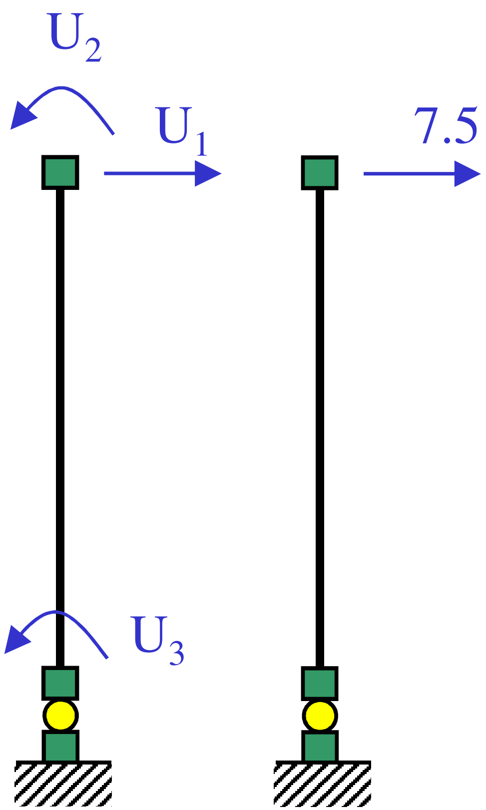




Example 2

PLASTIC HINGE 1: $\lambda_1 P = 7.5\ \text{kN}$

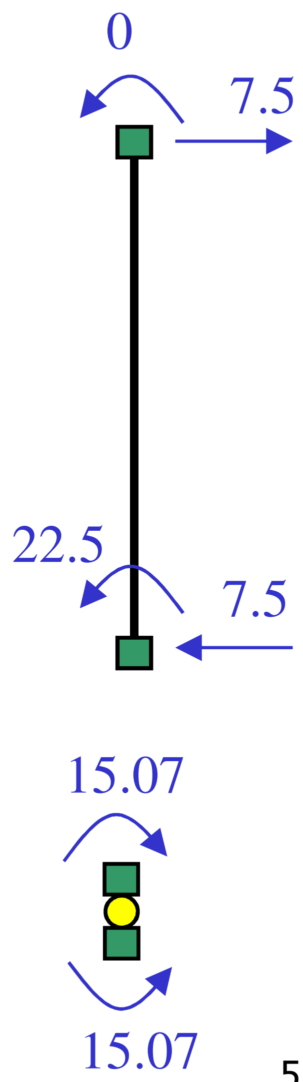
***i*=1**



$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 15.066586 \\ -15.066586 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 7.433414 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 7.433414 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -7.433414 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces
Apply $\mathbf{P}_{unb}$

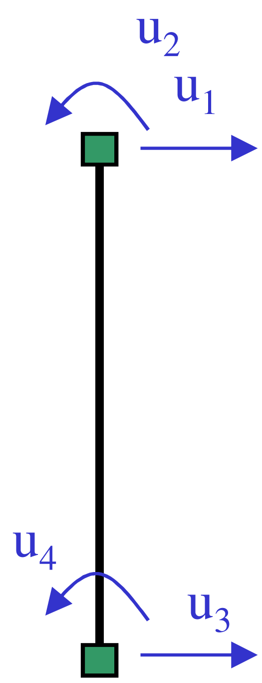
Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

i=2

$$\Delta \mathbf{U} = \mathbf{K}_0^{-1}\{\Delta \mathbf{P}\} = \begin{Bmatrix} 0.00044 \\ -0.00015 \\ -0.00015 \end{Bmatrix}$$

$$\mathbf{U} = \mathbf{U} + \Delta \mathbf{U} = \begin{Bmatrix} 0.00854 \\ -0.00397 \\ -0.000599 \end{Bmatrix}$$

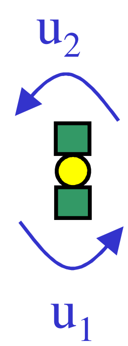


$$\mathbf{U}_b = \begin{Bmatrix} 0.00854 \\ -0.00397 \\ 0 \\ -0.000599 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.000599 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.000599$$



Example 2

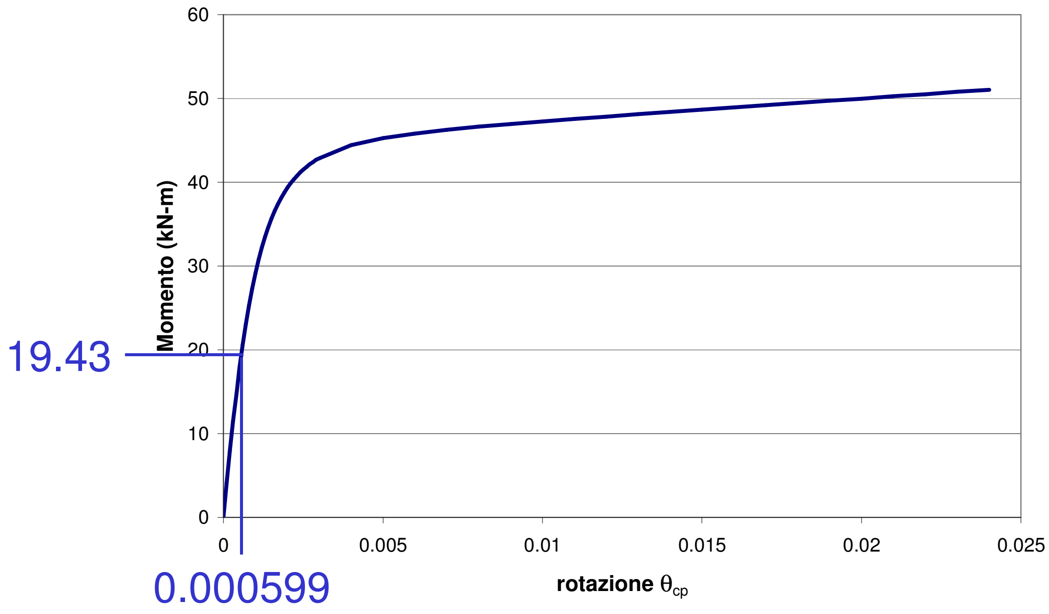
LOAD STEP 1: $\lambda_1 P = 7.5$ kN

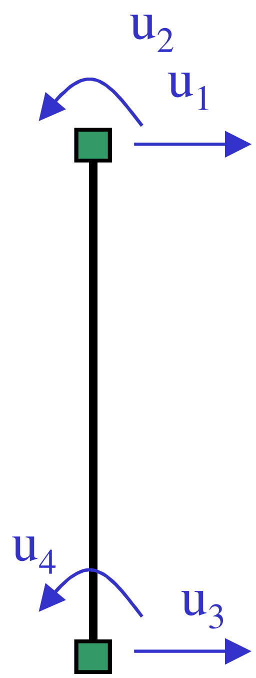
i=2

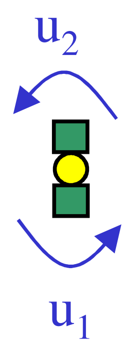
Elements' resisting forces

Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -19.43$ kN-m



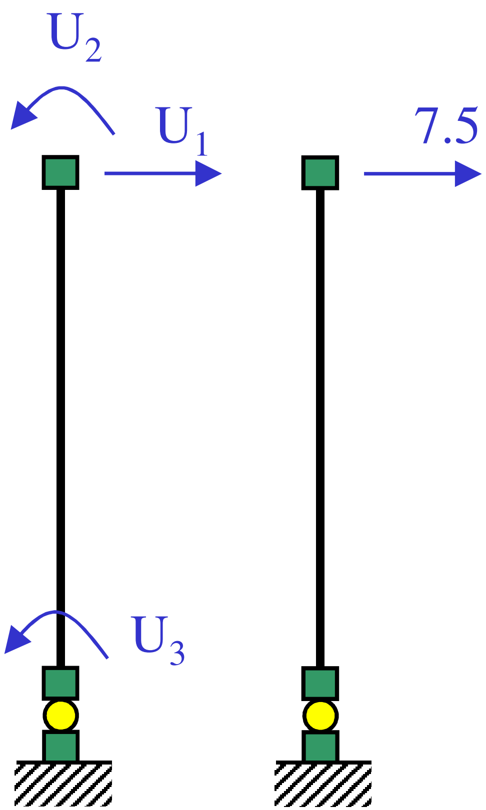




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

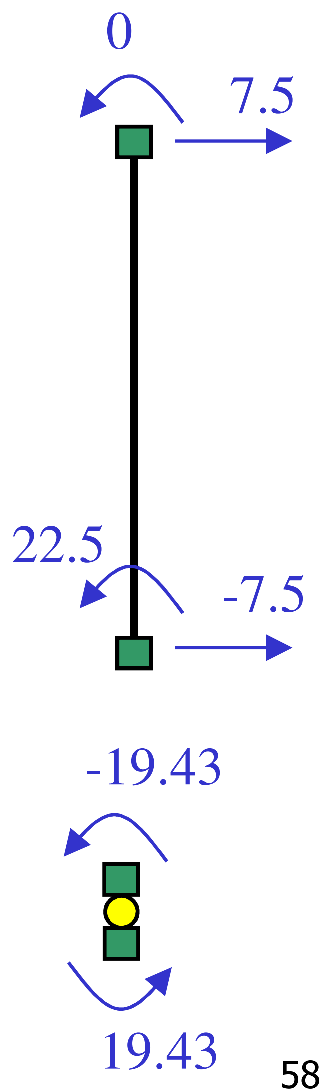
i=2



$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 19.430188 \\ -19.430188 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 3.069812 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 3.069812 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -3.069812 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces

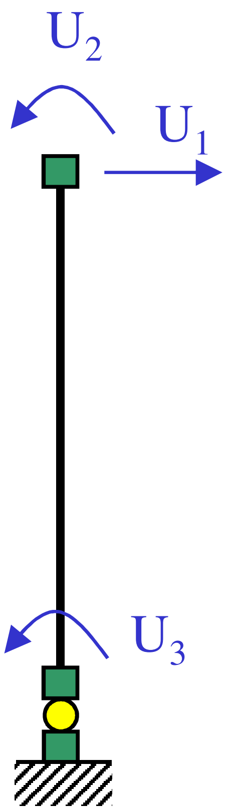
Apply $\mathbf{P}_{unb}$

Note that $\left\| \mathbf{P}_{unb}^{i=2} \right\| < \left\| \mathbf{P}_{unb}^{i=1} \right\|$

Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

$i=3$



$$\Delta \mathbf{U} = \mathbf{K}_0^{-1}\{\Delta \mathbf{P}\} = \begin{Bmatrix} 0.00018 \\ -0.00006 \\ -0.00006 \end{Bmatrix}$$

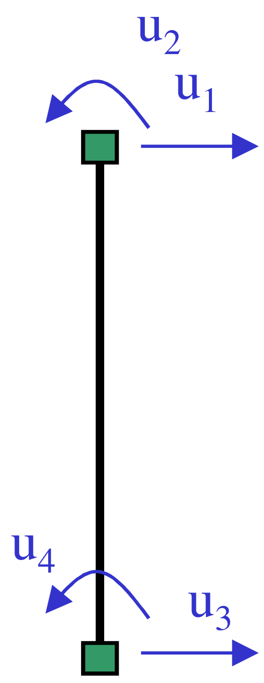
$$\mathbf{U} = \mathbf{U} + \Delta \mathbf{U} = \begin{Bmatrix} 0.00873 \\ -0.00403 \\ -0.00066 \end{Bmatrix}$$

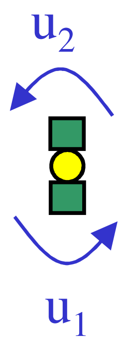
$$\mathbf{U}_b = \begin{Bmatrix} 0.00873 \\ -0.00403 \\ 0 \\ -0.00066 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00066 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.00066$$





Example 2

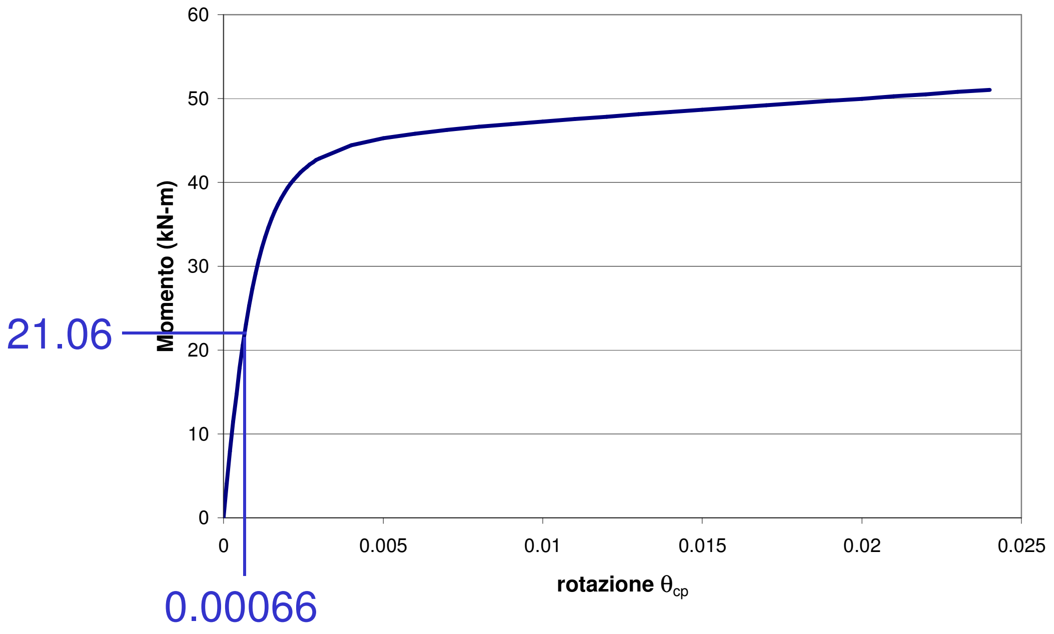
LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

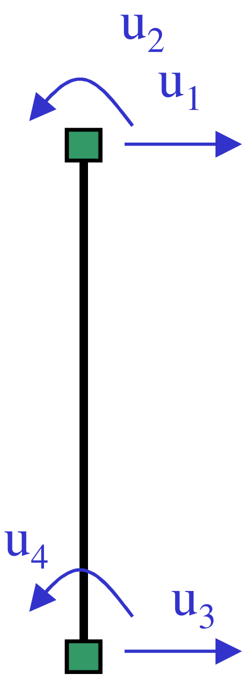
i=3

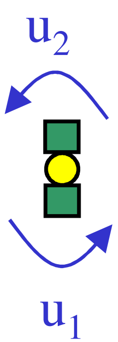
Elements' resisting forces

Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -21.06\ \text{kN-m}$



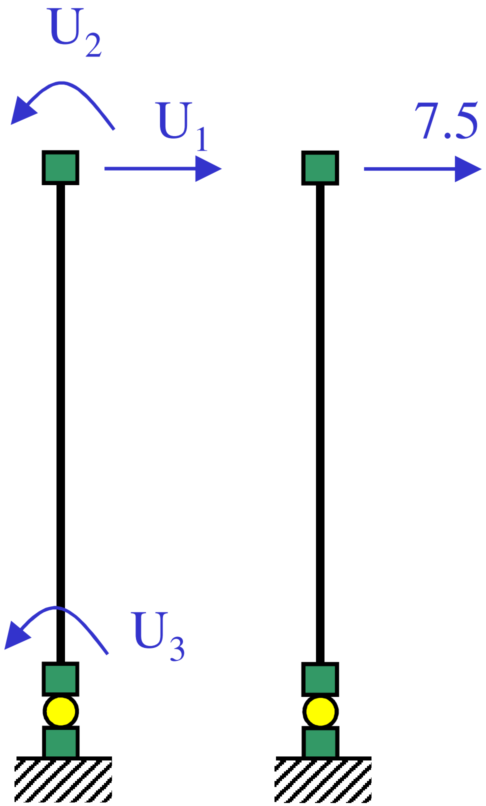




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

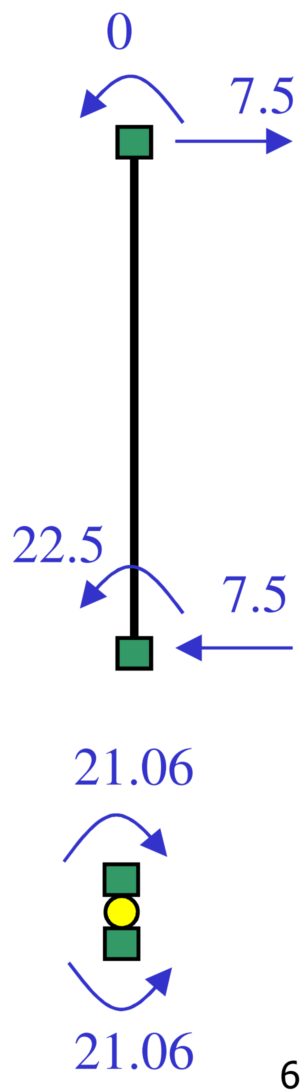
i=3



$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 21.060194 \\ -21.060194 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 1.439806 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 1.439806 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1.439806 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces
Apply $\mathbf{P}_{unb}$

Note that $\left\|\mathbf{P}_{unb}^{i=3}\right\| < \left\|\mathbf{P}_{unb}^{i=2}\right\|$

Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

***i*=15**

$$\Delta\mathbf{U} = \mathbf{K}_0^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.00000000007 \\ -0.00000000002 \\ -0.00000000002 \end{Bmatrix}$$

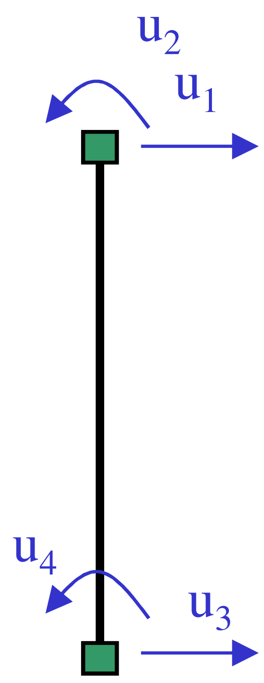
$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.0089 \\ -0.00409 \\ -0.00072 \end{Bmatrix}$$

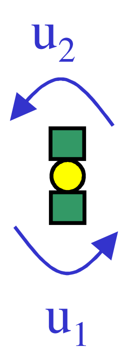
$$\mathbf{U}_b = \begin{Bmatrix} 0.0089 \\ -0.00409 \\ 0 \\ -0.00072 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00072 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.00072$$





Example 2

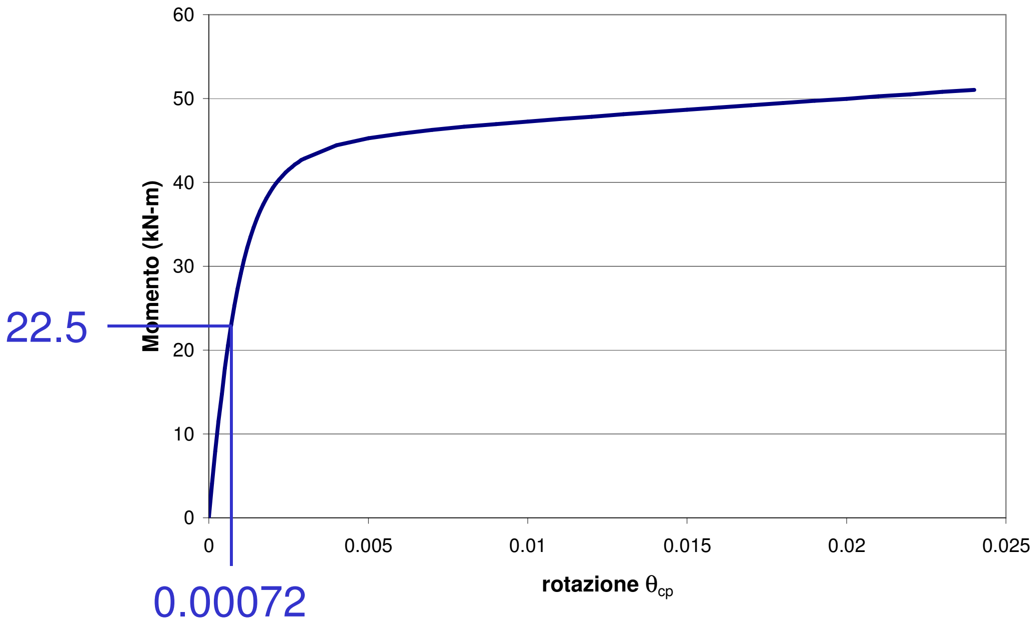
LOAD STEP 1: $\lambda_1 P = 7.5$ kN

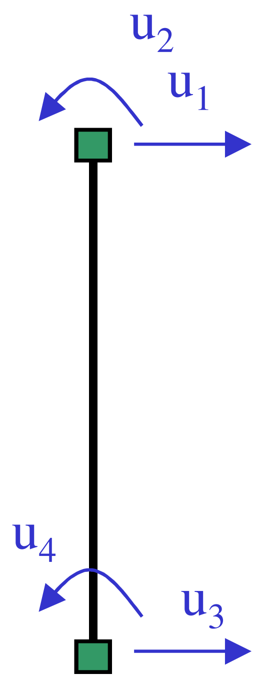
i=15

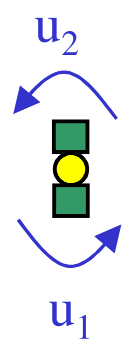
Elements' resisting forces

Column: linear elastic $\mathbf{P}_h = \mathbf{K}_h \mathbf{U}_h$

Plastic hinge $M_h = -22.5$ kN-m



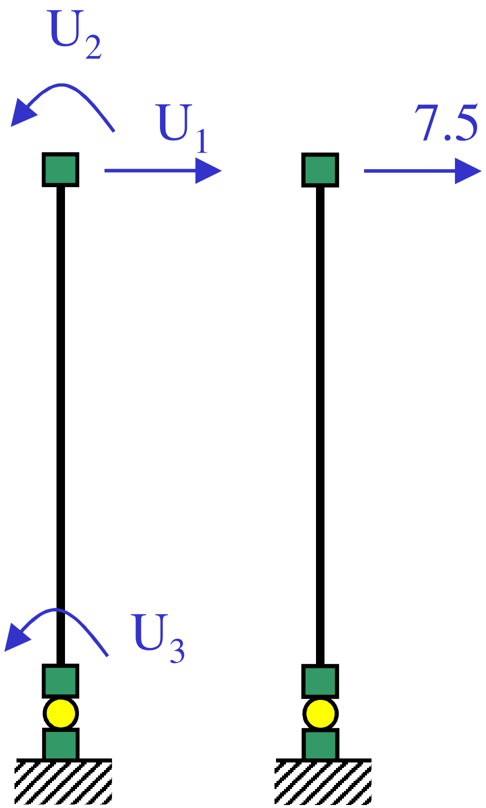




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

i=15

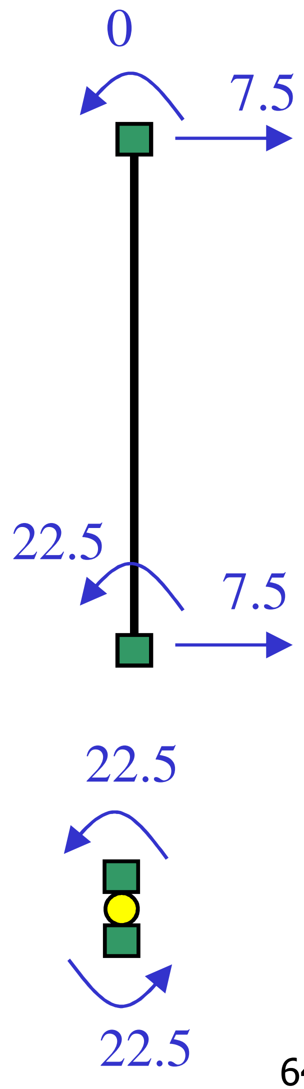


$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 22.499999 \\ -22.499999 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ .00000113 \end{Bmatrix}$$

$$\mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ .00000113 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -.00000113 \end{Bmatrix}$$

Small enough!!



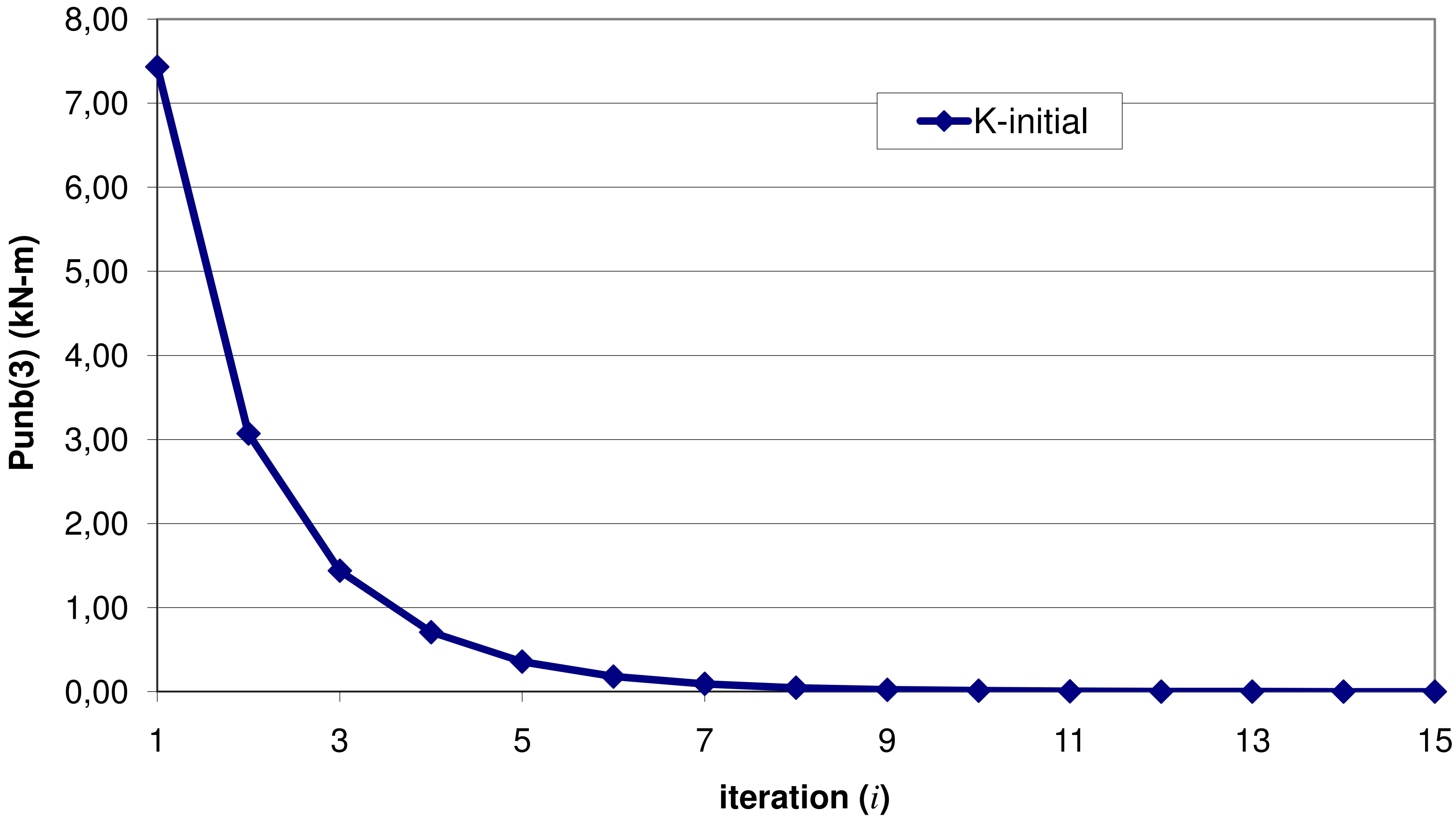
Note that $\left\| \mathbf{P}_{unb}^{i=15} \right\| \approx \mathbf{0}$

There is equilibrium between applied and resisting forces

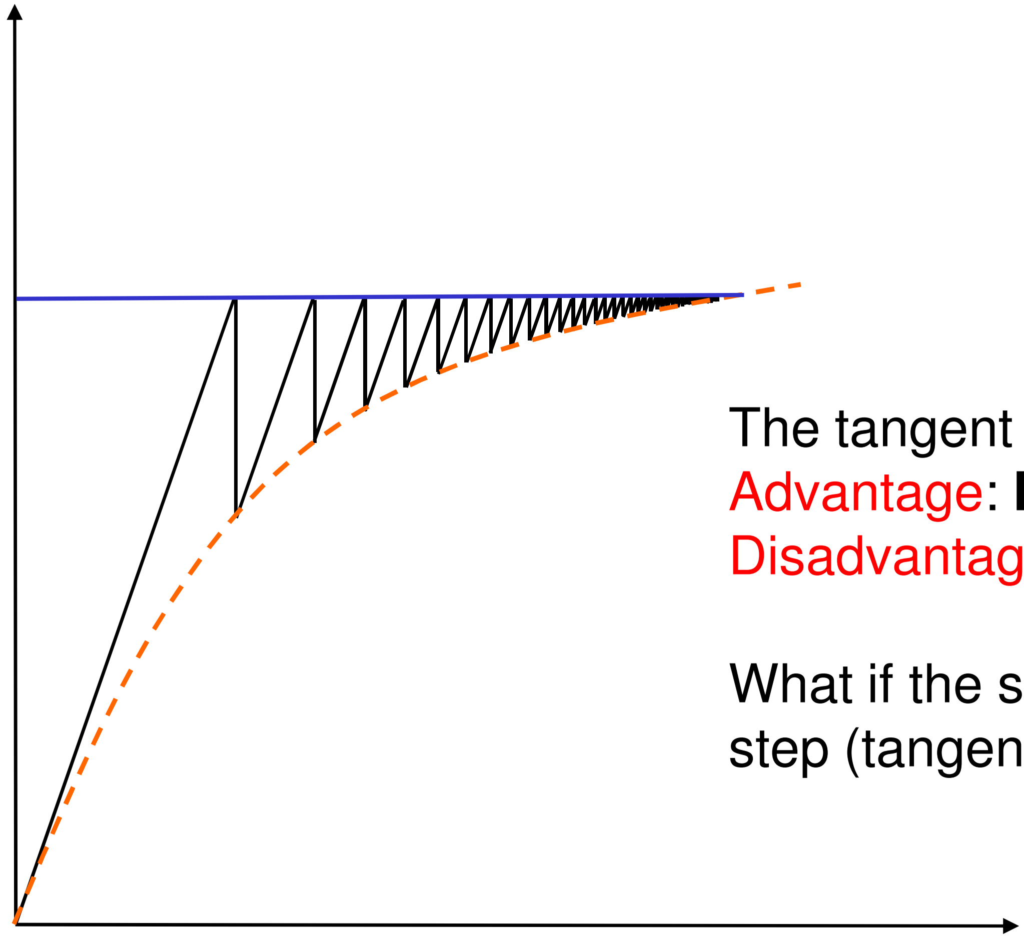
Apply $\lambda_2 P$

Example 2

Convergence was very slow because initial stiffness was used



Example 2

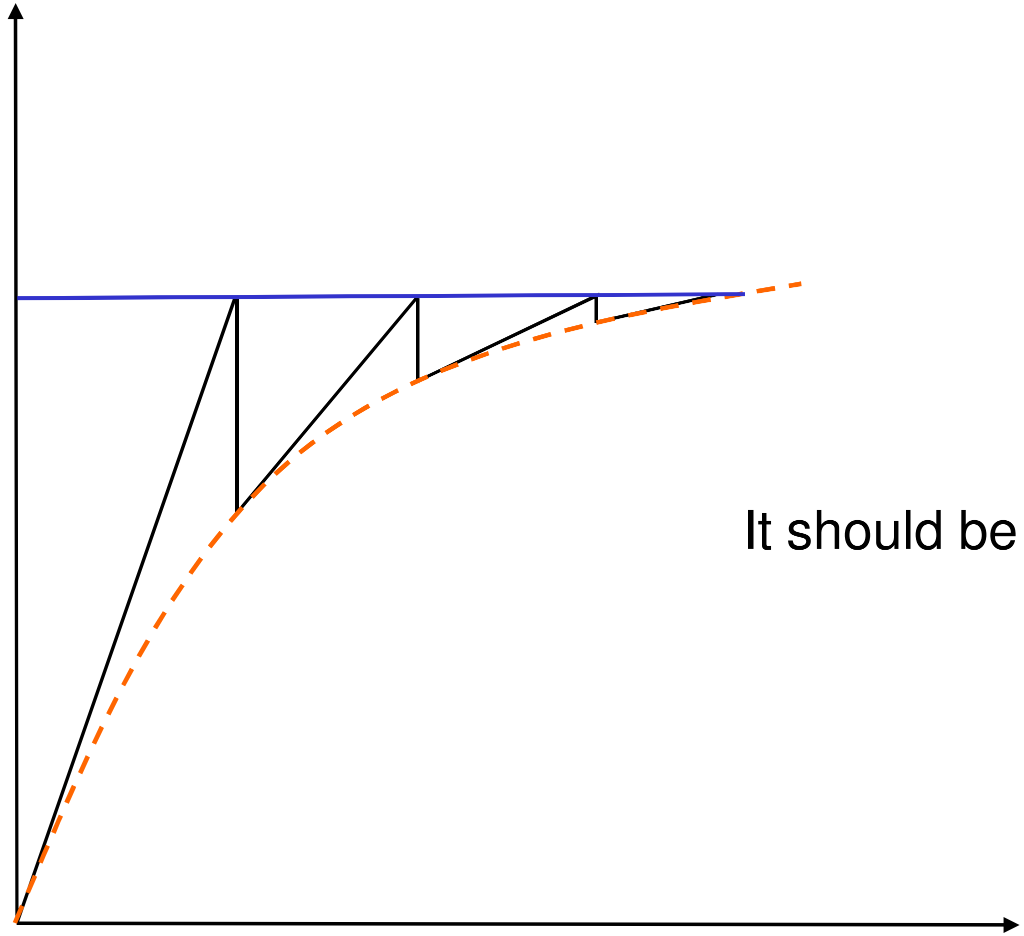


The tangent stiffness does not change:
Advantage: **K** is inverted only once
Disadvantage: convergence is slow

What if the stiffness is updated at every step (tangent stiffness)?

Example 2

Tangent stiffness (Newton-Raphson)



It should be much faster!

Example 2

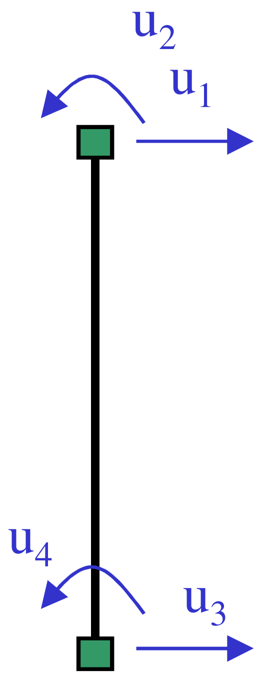
LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

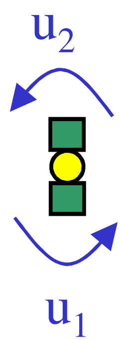
***i*=1**

$$\mathbf{U} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{P}_b = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{P}_h = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \mathbf{P}_R = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{P} = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix}$$





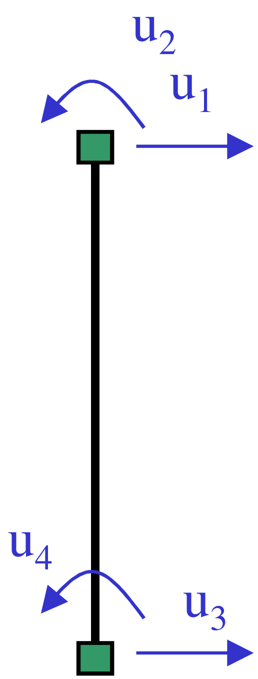
Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

***i*=1**

$$\Delta\mathbf{U} = \mathbf{K}_{tan}^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix} \qquad \text{for } i = 1, \quad \mathbf{K}_{tan} = \mathbf{K}_0$$

$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ -0.00045 \end{Bmatrix}$$

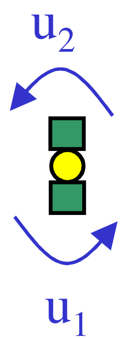


$$\mathbf{U}_b = \begin{Bmatrix} 0.0081 \\ -0.0038 \\ 0 \\ -0.00045 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00045 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.00045$$



Example 2

LOAD STEP 1: $\lambda_1 P = 7.5$ kN

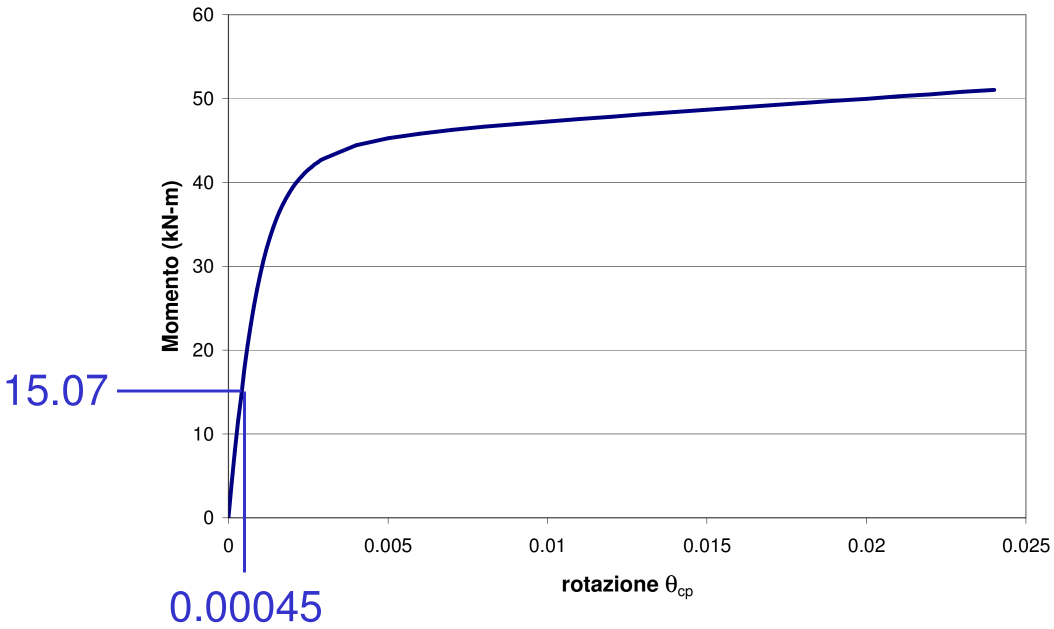
***i*=1**

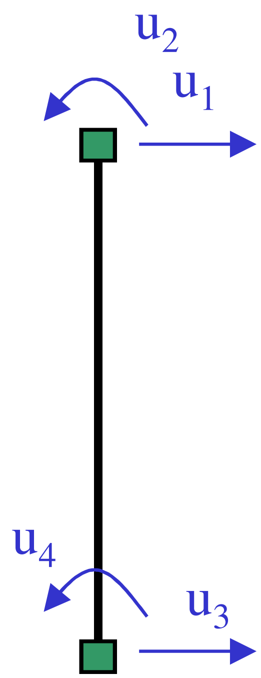
Elements' resisting forces

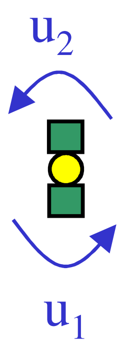
Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -15.07$ kN-m

$K_{h,tan} = 3.14\ 10^4$ kN-m



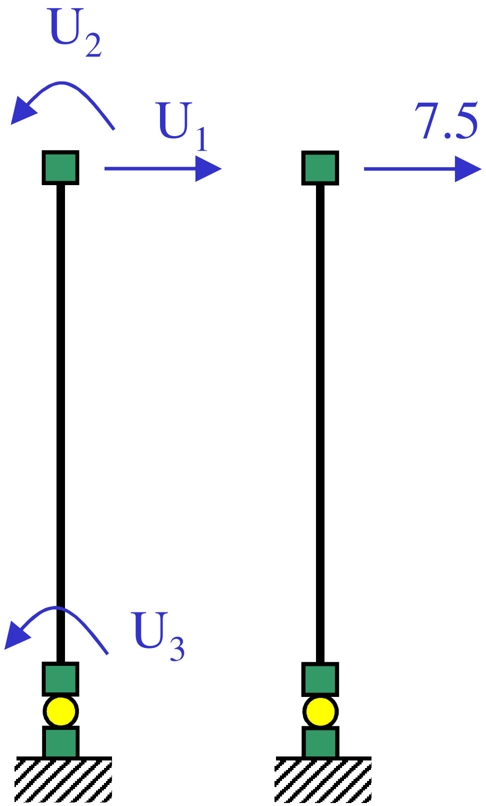




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

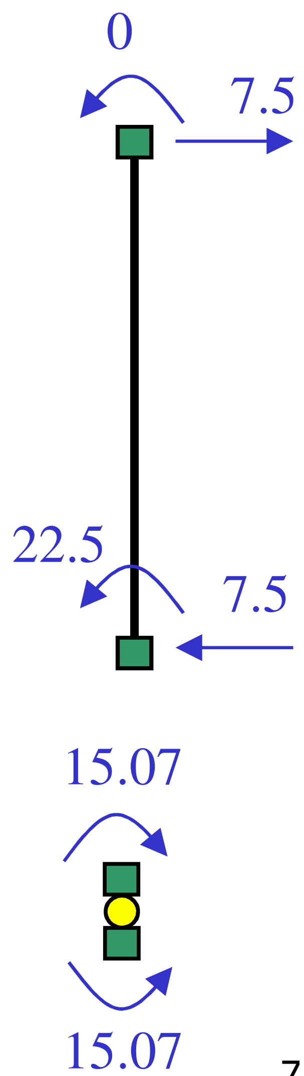
***i*=1**



$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 15.066586 \\ -15.066586 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 7.433414 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 7.433414 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -7.433414 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces

Apply $\mathbf{P}_{unb}$

Example 2

LOAD STEP 1: $\lambda_1 P = 7.5 \text{ kN}$

i=2

$$\Delta \mathbf{U} = \mathbf{K}_{\tan}^{-1} \{\Delta \mathbf{P}\} = \begin{Bmatrix} 0.00071 \\ -0.00024 \\ -0.00024 \end{Bmatrix}$$

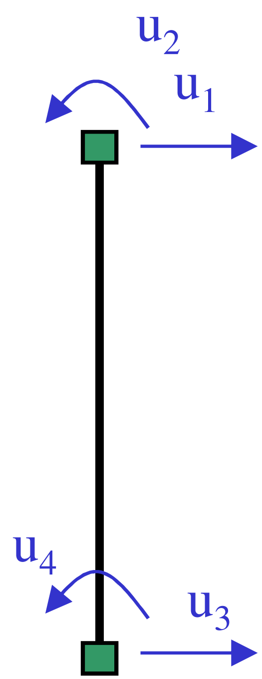
$$\mathbf{U} = \mathbf{U} + \Delta \mathbf{U} = \begin{Bmatrix} 0.00881 \\ -0.00406 \\ -0.000687 \end{Bmatrix}$$

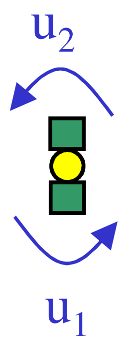
$$\mathbf{U}_b = \begin{Bmatrix} 0.00881 \\ -0.00406 \\ 0 \\ -0.000687 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.000687 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.000687$$





Example 2

LOAD STEP 1: $\lambda_1 P = 7.5$ kN

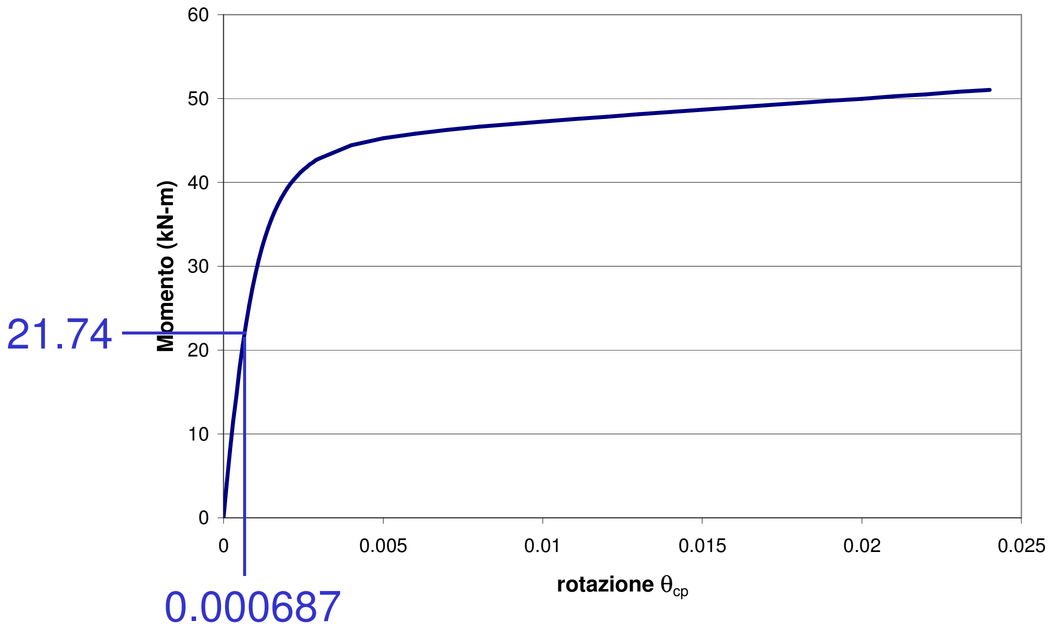
i=2

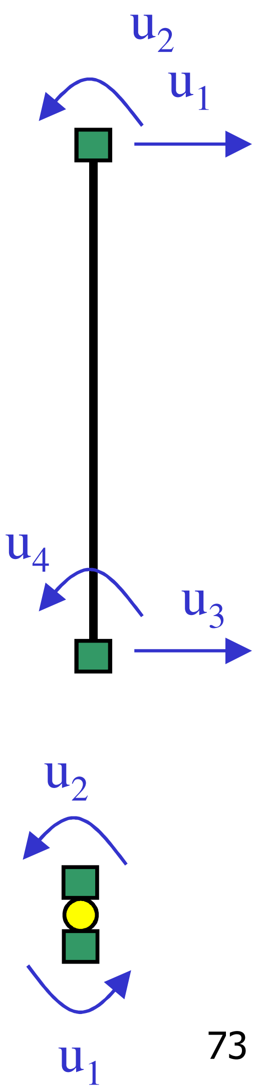
Elements' resisting forces

Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -21.74$ kN-m

$K_{h,tan} = 2.5\ 10^4$ kN-m

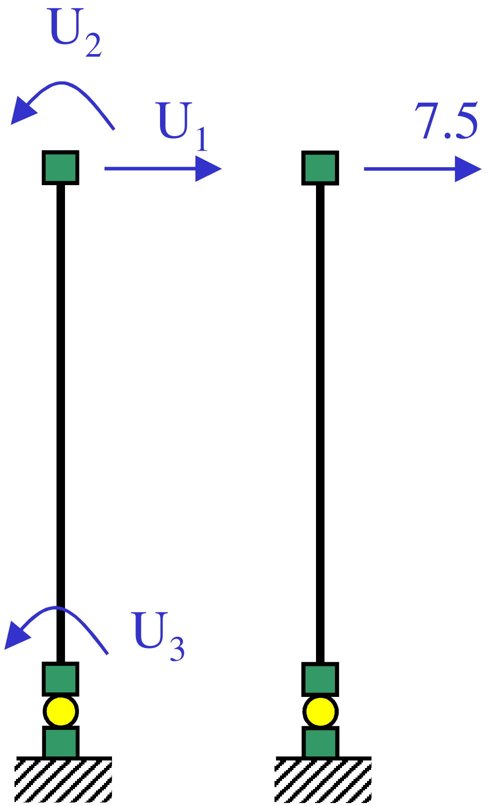




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

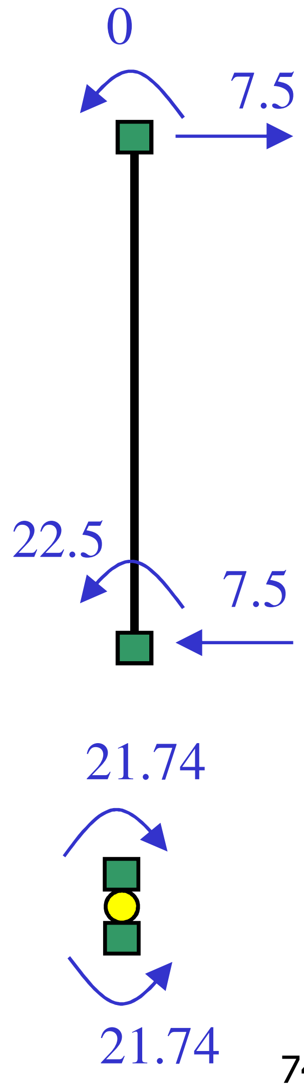
***i*=2**



$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 21.743466 \\ -21.743466 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0.756533 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 0.756533 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.756533 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces

Apply P_{unb}

Note that $\left\| \mathbf{P}_{unb}^{i=2} \right\| \ll \left\| \mathbf{P}_{unb}^{i=1} \right\|$

Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

i=3

$$\Delta\mathbf{U} = \mathbf{K}_{\tan}^{-1}\{\Delta\mathbf{P}\} = \begin{Bmatrix} 0.00009 \\ -0.00003 \\ -0.00003 \end{Bmatrix}$$

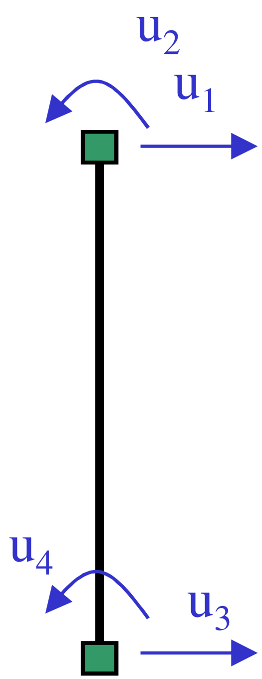
$$\mathbf{U} = \mathbf{U} + \Delta\mathbf{U} = \begin{Bmatrix} 0.0089 \\ -0.00409 \\ -0.000717 \end{Bmatrix}$$

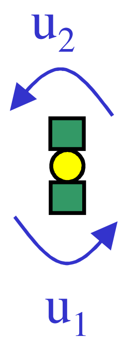
$$\mathbf{U}_b = \begin{Bmatrix} 0.0089 \\ -0.00400 \\ 0 \\ -0.000717 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.000717 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.000717$$





Example 2

LOAD STEP 1: $\lambda_1 P = 7.5$ kN

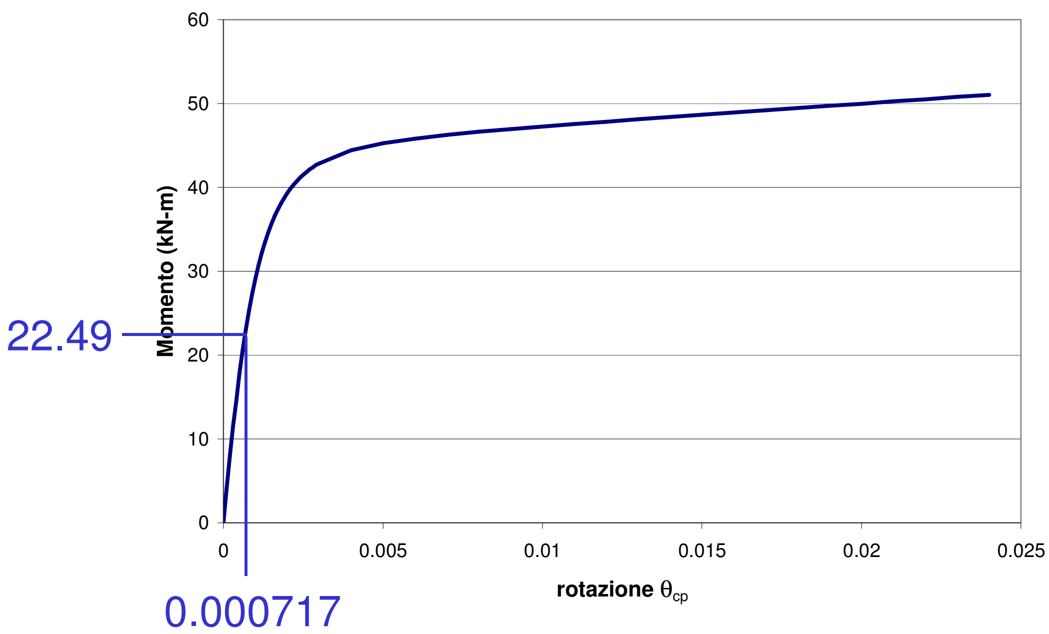
***i*=3**

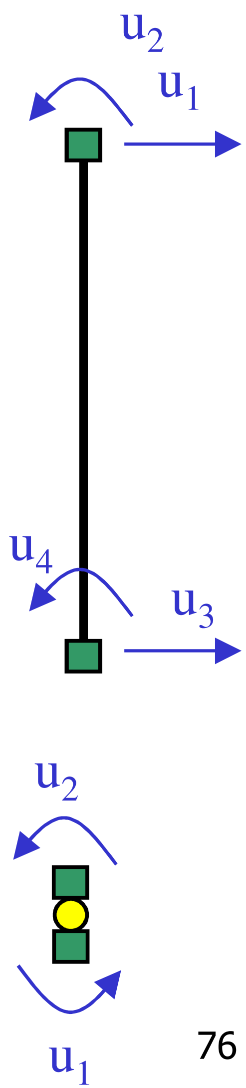
Elements' resisting forces

Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -22.49$ kN-m

$K_{h,tan} = 2.43\ 10^{10}$ N-mm

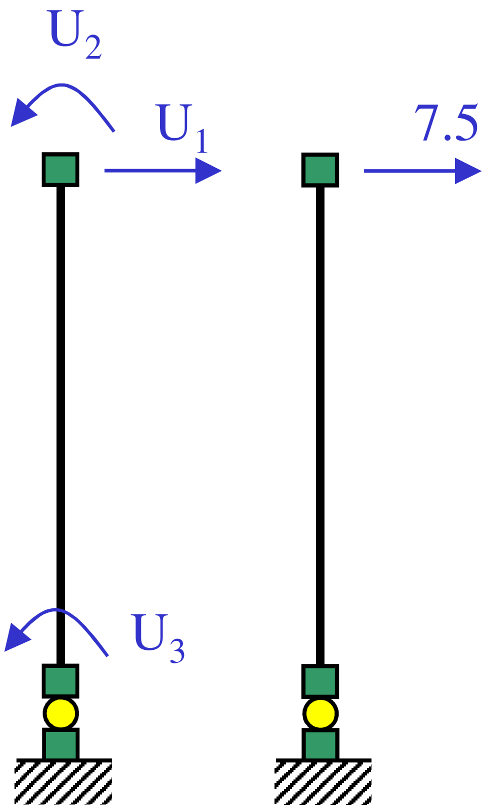




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5 \text{ kN}$

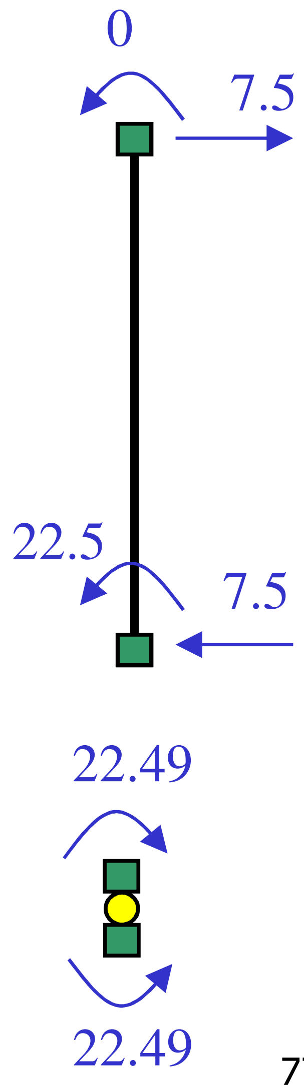
i=3



$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 22.488482 \\ -22.488482 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} -7.5 \\ 0 \\ -0.011518 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 0.011518 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.011518 \end{Bmatrix}$$



There is no equilibrium between applied and resisting forces
Apply $\mathbf{P}_{unb}$

Note that $\left\|\mathbf{P}_{unb}^{i=3}\right\| \ll \left\|\mathbf{P}_{unb}^{i=2}\right\|$

Example 2

LOAD STEP 1: $\lambda_1 P = 7.5 \text{ kN}$

***i*=4**

$$\mathbf{\Delta U} = \mathbf{K}_{\tan}^{-1}\{\mathbf{\Delta P}\} = \begin{Bmatrix} 0.0000014 \\ -0.000000474 \\ -0.000000474 \end{Bmatrix}$$

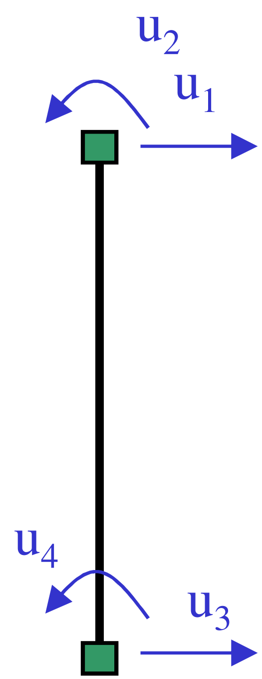
$$\mathbf{U} = \mathbf{U} + \mathbf{\Delta U} = \begin{Bmatrix} 0.0089 \\ -0.00409 \\ -0.00072 \end{Bmatrix}$$

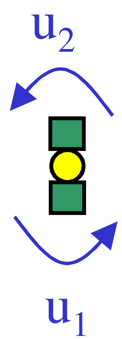
$$\mathbf{U}_b = \begin{Bmatrix} 0.0089 \\ -0.00409 \\ 0 \\ -0.00072 \end{Bmatrix}$$

$$\mathbf{U}_h = \begin{Bmatrix} 0 \\ -0.00072 \end{Bmatrix}$$

$$\Downarrow$$

$$\theta_h = -0.00072$$





Example 2

LOAD STEP 1: $\lambda_1 P = 7.5 \text{ kN}$

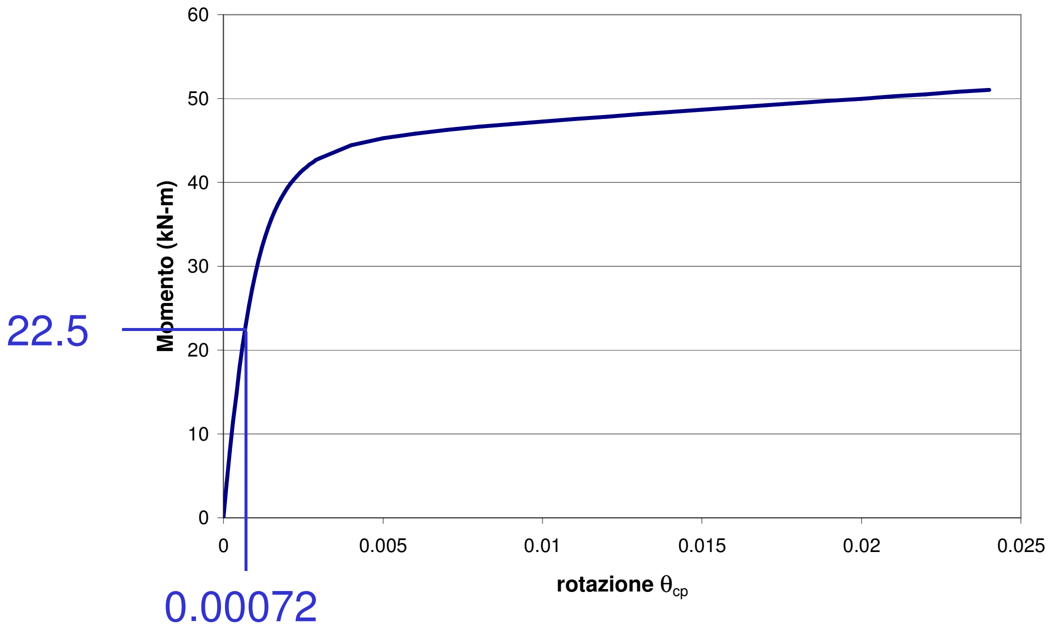
***i*=4**

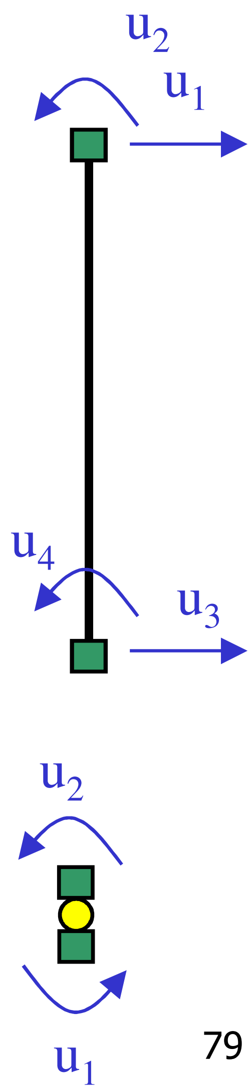
Elements' resisting forces

Column: linear elastic $\mathbf{P}_b = \mathbf{K}_b \mathbf{U}_b$

Plastic hinge $M_h = -22.5 \text{ kN-m}$

$K_{h,tan} = 2.4289\ 10^4 \text{ kN-m}$

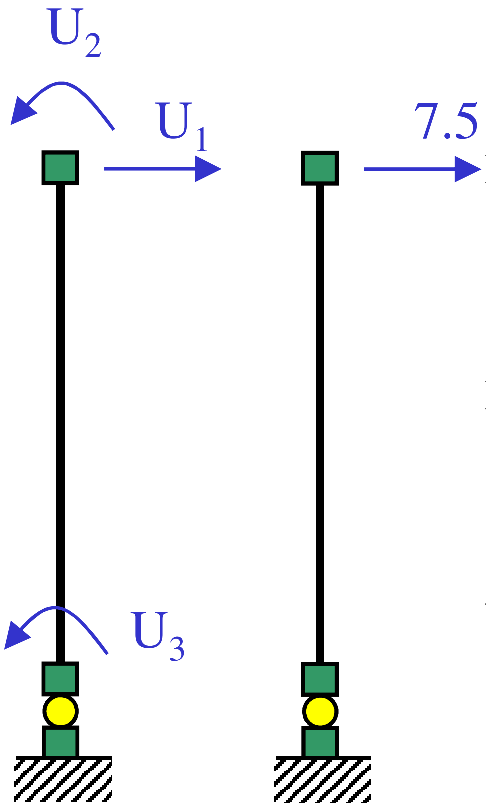




Example 2

LOAD STEP 1: $\lambda_1 P = 7.5\ \text{kN}$

i=4

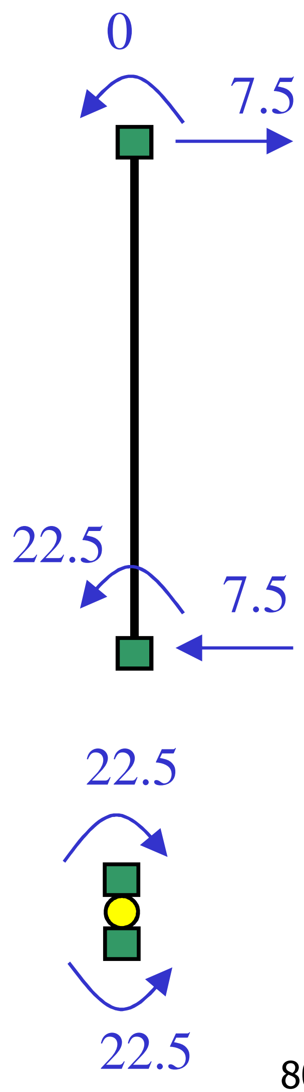


$$\mathbf{P}_b = \begin{Bmatrix} 7.5 \\ 0 \\ -7.5 \\ 22.5 \end{Bmatrix} \qquad \mathbf{P}_h = \begin{Bmatrix} 22.499999 \\ -22.499999 \end{Bmatrix}$$

$$\mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0.0000028 \end{Bmatrix}$$

$$\Delta\mathbf{P} = \mathbf{P}_{unb} = \mathbf{P} - \mathbf{P}_R = \begin{Bmatrix} 7.5 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 7.5 \\ 0 \\ 0.0000028 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.0000028 \end{Bmatrix}$$

Small enough!!



Note that $\left\|\mathbf{P}_{unb}^{i=4}\right\| \ll \left\|\mathbf{P}_{unb}^{i=3}\right\|$

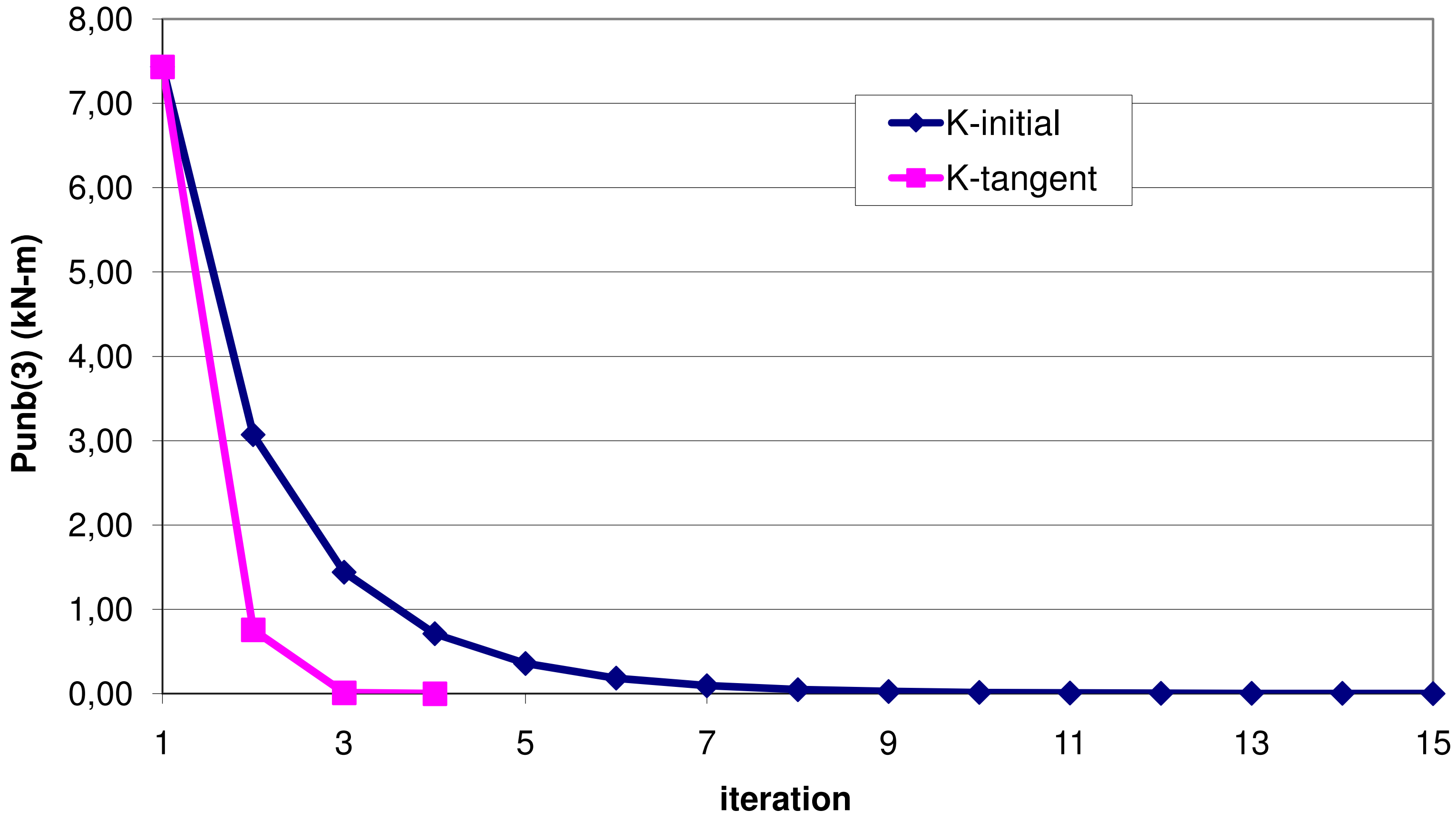
$$\left\|\mathbf{P}_{unb}^{i=4}\right\| \approx \mathbf{0}$$

There is equilibrium between applied and resisting forces

Apply $\lambda_2 P$

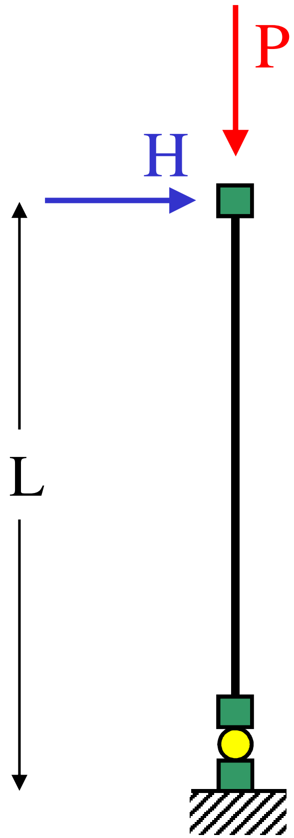
Example 2

COMPARISON BETWEEN CONVERGENCE SPEEDS



Example 3

NONLINEAR GEOMETRY: P-Δ EFFECT



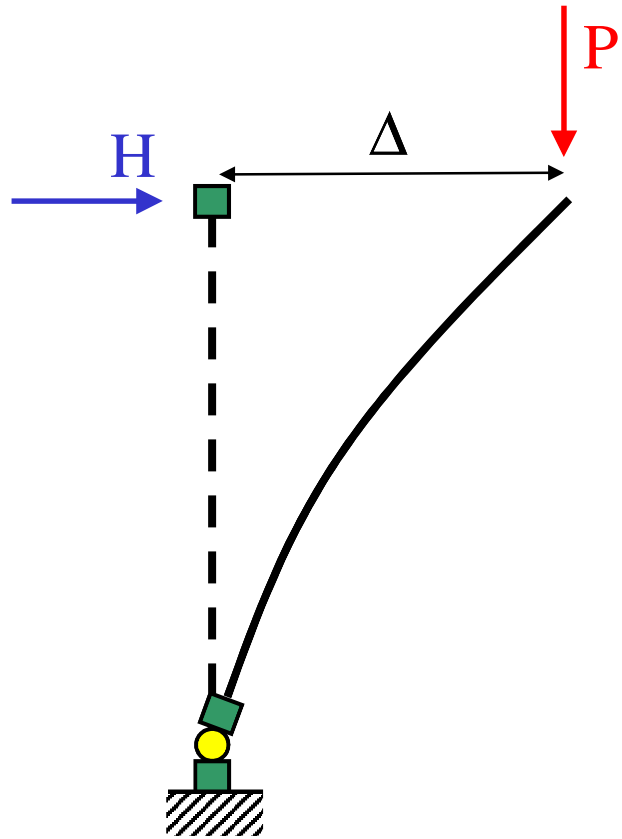
$$P\Delta << HL$$

Equilibrium
in the undeformed configuration

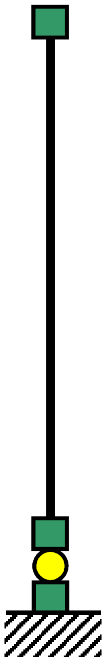
$$M_{base} = HL$$

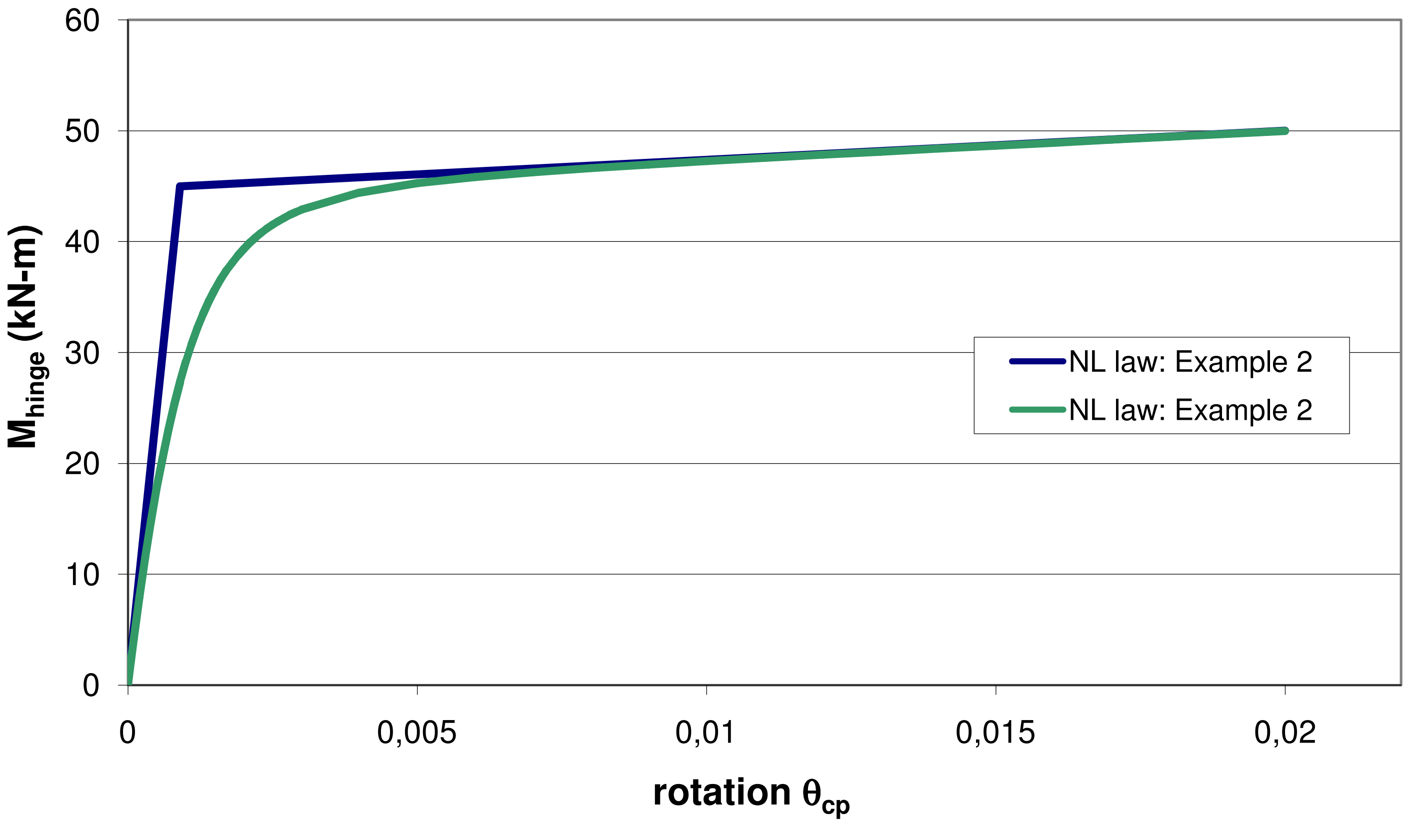
otherwise
Equilibrium
in the deformed configuration

$$M_{base} = HL + P\Delta$$



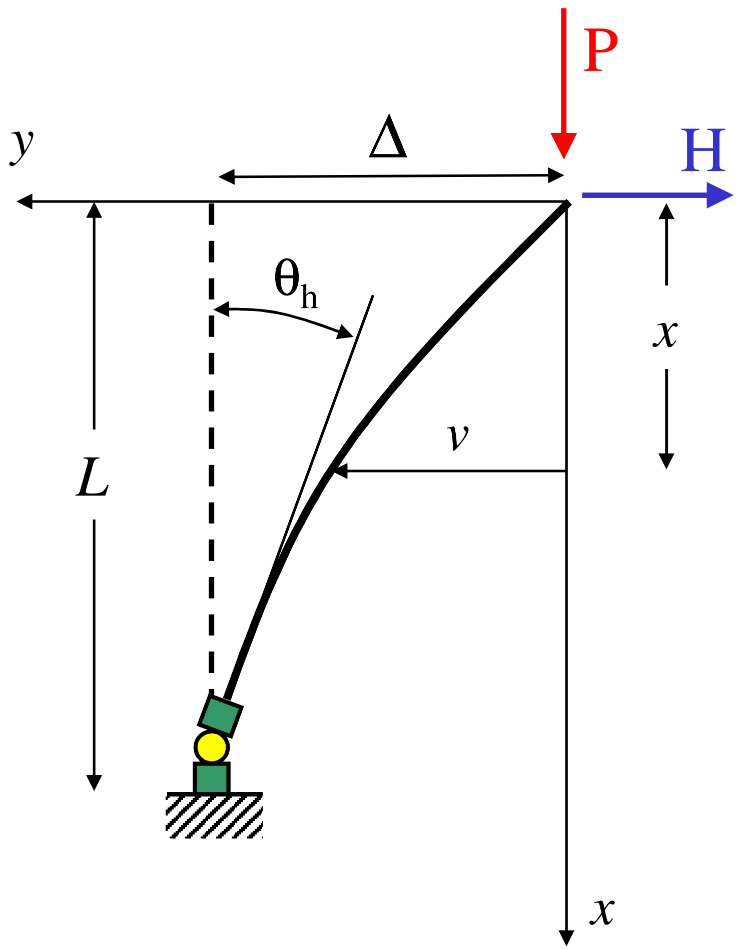
Example 3





Nonlinear hinge law: from Example 2

Example 3



$$M(x) = -Pv - Hx$$

$$EIv'' = -Pv - Hx$$

$$v'' + \frac{P}{EI}v = -\frac{Hx}{EI}$$

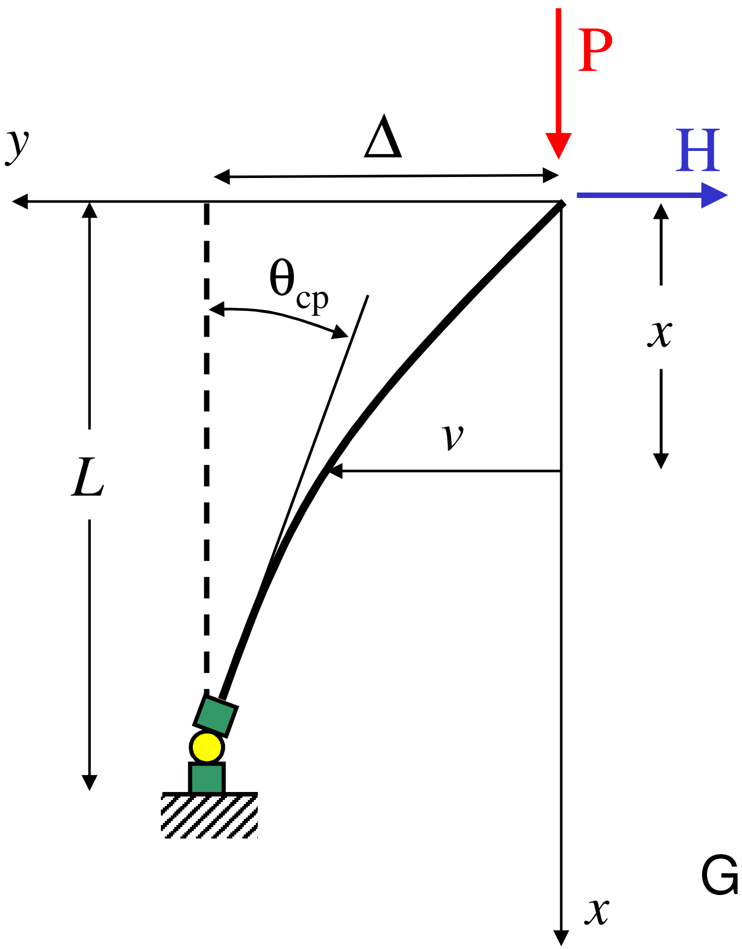
$$v(x) = C_1 \sin\sqrt{\frac{P}{EI}}x + C_2 \cos\sqrt{\frac{P}{EI}}x - \frac{H}{P}x$$

$$v(0) = 0 \quad \Rightarrow \quad C_2 = 0$$

$$v'(L) = \theta_h \quad \Rightarrow \quad C_1 = \frac{H/P + \theta_h}{\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}L}$$

Example 3

EQUILIBRIUM IN THE DEFORMED CONFIGURATION



$$v(x) = \frac{H/P + \theta_h}{\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}L}\sin\sqrt{\frac{P}{EI}}x - \frac{H}{P}x$$

$$\Delta = v(L) = \frac{H}{P}\frac{\tan\sqrt{\frac{P}{EI}}L}{\sqrt{\frac{P}{EI}}} + \frac{\theta_h}{\sqrt{\frac{P}{EI}}}\tan\sqrt{\frac{P}{EI}}L - \frac{H}{P}L$$

$$\textit{from equilibrium} \quad M_h = HL + P\Delta$$

$$\Delta = \frac{M_h - HL}{P}$$

Get closed form solution Δ

Example 3

EQUILIBRIUM IN THE DEFORMED CONFIGURATION

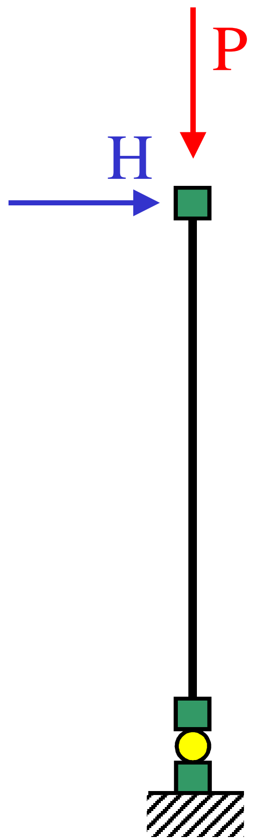
$$M_h = H \frac{\tan\sqrt{\frac{P}{EI}}L}{\sqrt{\frac{P}{EI}}} + P \frac{\theta_h}{\sqrt{\frac{P}{EI}}} \tan\sqrt{\frac{P}{EI}}L$$

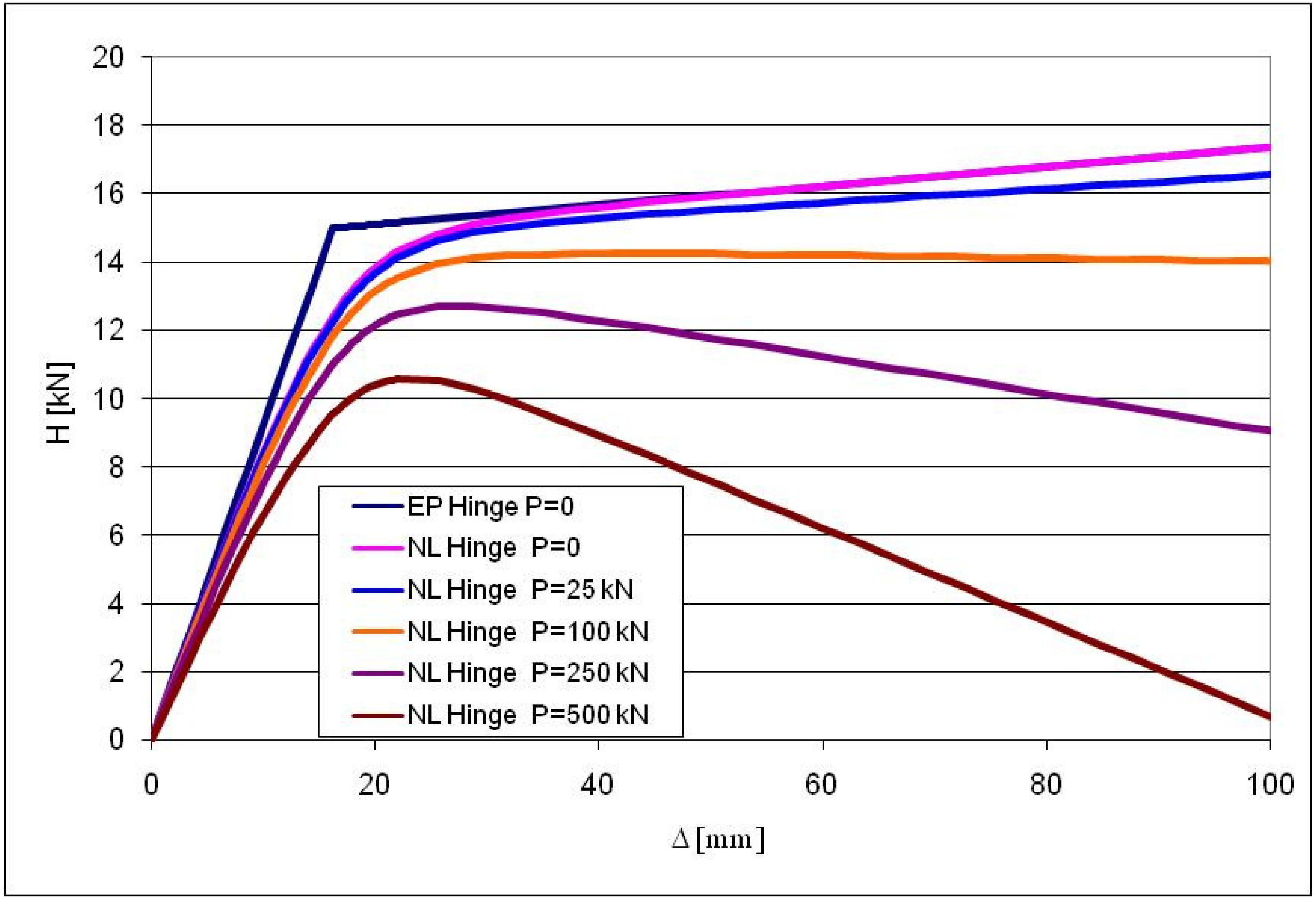
$$H = M_h \frac{\sqrt{\frac{P}{EI}}}{\tan\sqrt{\frac{P}{EI}}L} - P\theta_h$$

$$\Delta = \frac{M_h - HL}{P}$$

Assign $\theta_h \Rightarrow M_h \Rightarrow H \Rightarrow \Delta$

Example 3

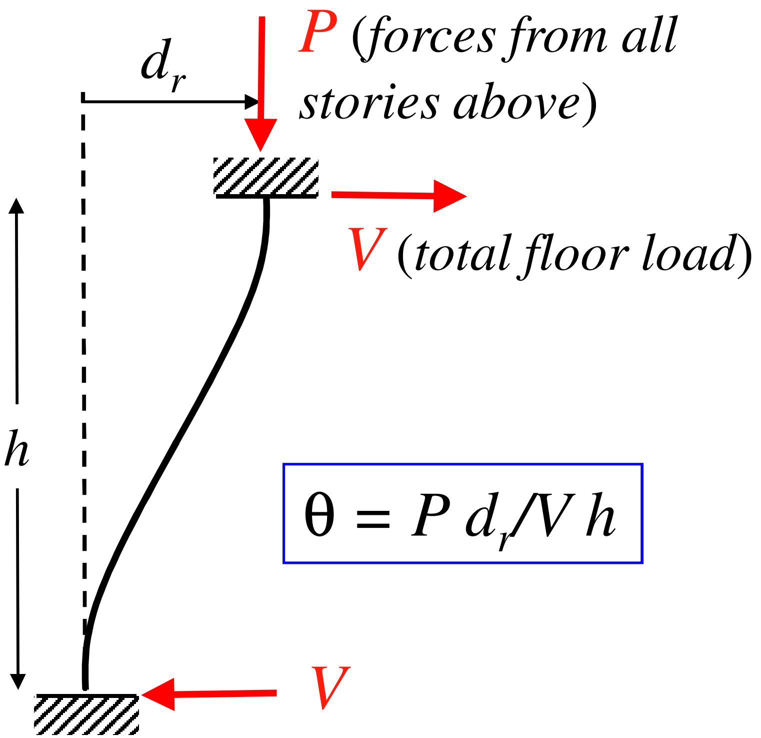




$$P_{cr\text{-}el} = \frac{\pi^2 EI}{4L^2} = 2740\,\text{kN}$$

Nonlinear geometry in NTC 2008

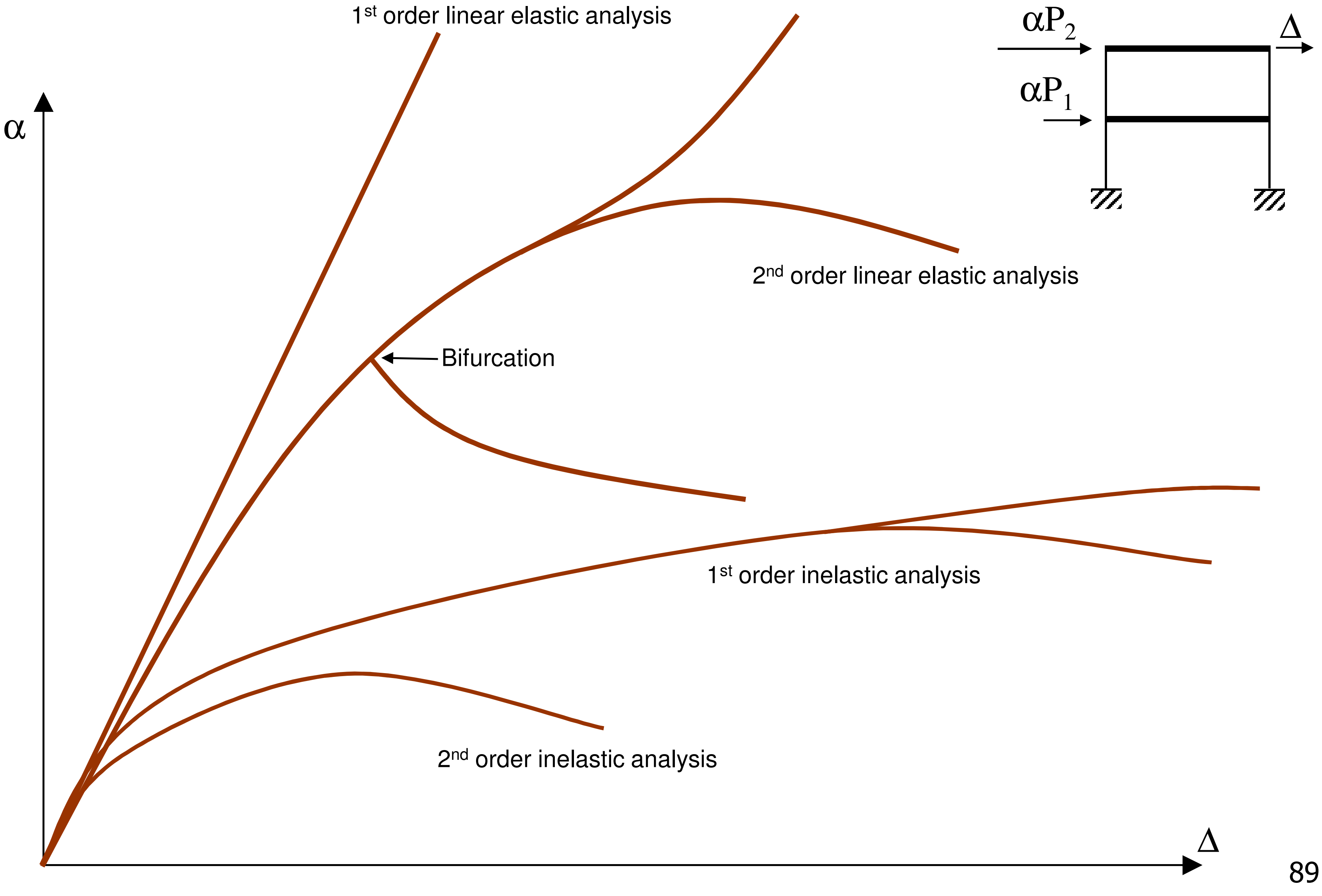
- 7.3.1 ... Second order effests



$$\theta = P\, d_r / V\, h$$

$\theta < 0{,}1$	Secondo order effects are neglected
$0{,}1 < \theta < 0{,}2$	Horizontal seismic action Effects are incremented by $1/(1-\theta)$
$0{,}2 < \theta < 0{,}3$	No comment
$\theta > 0{,}3$	Not allowed

Conclusions



Conclusions

Elastic Analysis – Materials are all linear elastic
Inelastic Analysis – Materials are inelastic

} Material

First order analysis – Equilibrium in the underformed configuration
Second order analysis – Equilibrium in the deformed configuration (large displacements, small, moderate or finite deformations)

} Geometry

Structural collapse is typically associated with loads that lead materials into the inelastic range, and with displacements that lead to structural instability at collapse